

**Lecture 4: 2.2 Cont..** The **arc length** of a curve from  $\mathbf{R}(a)$  to  $\mathbf{R}(t)$  is given by

$$(2.2.12) \quad s = s(t) = \int_a^t \left| \frac{d\mathbf{R}}{dt}(\tau) \right| d\tau,$$

By the Fundamental Theorem of Calculus

$$(2.2.13) \quad \frac{ds(t)}{dt} = \left| \frac{d\mathbf{R}}{dt}(t) \right|.$$

If the curve describes the motion of a particle then (2.2.13) says that the change of the distance per unit time is the speed. It is often useful to reparametrize the curve by arc length because it does not depend on any particular parametrization so it is better to use if we want to study properties of the curve itself. If  $t(s)$  is the inverse of the function in (2.12.2) then

$$(2.2.14) \quad \mathbf{R}(t(s))$$

is a reparametrization of the curve and by the chain rule and (2.2.13)

$$(2.2.15) \quad \frac{d\mathbf{R}(t(s))}{ds} = \frac{d\mathbf{R}(t)}{dt} \frac{dt}{ds} = \frac{\mathbf{R}'(t)}{|\mathbf{R}'(t)|}, \quad \text{so} \quad \left| \frac{d\mathbf{R}(t(s))}{ds} \right| = 1.$$

**Ex.** Parametrize the helix  $\mathbf{R}(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + t \mathbf{k}$  with respect to the arc length.

**Sol.** We have  $\mathbf{R}'(t) = -\sin t \mathbf{i} + \cos t \mathbf{j} + \mathbf{k}$  so  $ds/dt = |\mathbf{R}'(t)| = \sqrt{2}$  and  $s(t) = \int_0^t |\mathbf{R}'(\tau)| d\tau = \int_0^t \sqrt{2} d\tau = \sqrt{2}t$ , so  $t(s) = s/\sqrt{2}$ . Hence the curve reparametrized in terms of arc length is  $\mathbf{R}(t(s)) = \cos(s/\sqrt{2}) \mathbf{i} + \sin(s/\sqrt{2}) \mathbf{j} + (s/\sqrt{2}) \mathbf{k}$ .

**Remark about notation.** In an example in the book where they reparametrize a curve  $\mathbf{R}(t)$  they have written  $\mathbf{R}(s)$  for the function we have denoted by  $\mathbf{R}(t(s))$ . We will try to avoid doing this since it can be misunderstood. This convention comes from Physics, where one thinks of the physical quantities as given:  $\mathbf{R}$  is the coordinate along the curve,  $t$  is the time and  $s$  is the arc length. Writing  $\mathbf{R}(t)$  or  $\mathbf{R}(s)$  in physics just means that the point depends on  $t$  or  $s$ . However, writing  $\mathbf{R}(t)$  in mathematics means that  $\mathbf{R}(t)$  is a given function of  $t$  and  $\mathbf{R}(s)$  would not mean the same thing as  $\mathbf{R}(t(s))$ . The problem really shows up at the level of derivative. What is  $\mathbf{R}'(s)$  supposed to mean? Is it  $\mathbf{R}'(t(s))$  or is it  $d\mathbf{R}(s)/ds = \mathbf{R}'(t(s))t'(s)$ ?

**Section 2.3: Curvature.** The tangent line to a curve is the line that best approximates the curve close to a point. The tangent line to a curve is the limit of the lines through two points  $\mathbf{R}(t)$  and  $\mathbf{R}(t+h)$  as  $h \rightarrow 0$ . If we want to approximate a curve better close to a point the question is what is the circle that best approximates a curve close to a point? This circle, called the **osculating circle**, is the limit of the circles through three points  $\mathbf{R}(t)$ ,  $\mathbf{R}(t+h_1)$  and  $\mathbf{R}(t+h_2)$  as  $h_1, h_2 \rightarrow 0$ . It is the circle that curves as much as the curve at the point. We want to measure how much a curve curves. Recall that  $\mathbf{R}'(t)$  is a tangent vector to the curve and

$$(2.3.1) \quad \mathbf{T}(t) = \frac{\mathbf{R}'(t)}{|\mathbf{R}'(t)|} = \frac{d\mathbf{R}(t(s))}{ds}$$

is a unit vector in the direction of the tangent line to the curve. If the curve is parametrized by arc length  $s$  then there is no need to divide by  $|\mathbf{R}'(t)|$ . Note that

$\mathbf{T}(t)$  changes direction very slowly when the curve is fairly straight. The rate of change of  $\mathbf{T}(t)$  per unit distance along the curve, therefore tells us something about how much the curve bends. We therefore define the **curvature** of a curve to be:

$$(2.3.2) \quad k = \left| \frac{d\mathbf{T}}{ds} \right|$$

Note that the derivative is defined in terms of the arc length  $s$  along the curve, i.e. we measure the rate of change of  $\mathbf{T}$  per unit distance traveled along the curve. It is easier to compute the curvature if it is expressed in terms of  $t$  instead of  $s$ . By the chain rule  $d\mathbf{T}/dt = d\mathbf{T}/ds ds/dt$  and by (2.2.13)  $ds/dt = |\mathbf{R}'(t)|$  so

$$(2.3.3) \quad k(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{R}'(t)|}$$

**Ex.** Show that the curvature of a circle of radius  $\rho$  is  $1/\rho$ .

**Sol.** If  $\mathbf{R}(t) = \rho \cos t \mathbf{i} + \rho \sin t \mathbf{j}$ , then  $\mathbf{R}'(t) = -\rho \sin t \mathbf{i} + \rho \cos t \mathbf{j}$  and  $|\mathbf{R}'(t)| = \rho$ . If  $\mathbf{T}(t) = \mathbf{R}'(t)/|\mathbf{R}'(t)| = -\sin t \mathbf{i} + \cos t \mathbf{j}$  then  $\mathbf{T}'(t) = -\cos t \mathbf{i} - \sin t \mathbf{j}$  and therefore  $k(t) = |\mathbf{T}'(t)|/|\mathbf{R}'(t)| = 1/\rho$ .

**The principal normal.** The geometric interpretation of curvature is that it is one over the radius of the osculating circle that best approximates the curve close to a point. The osculating circle lies in a plane called the osculating plane. Let  $\mathbf{N}$  denote a unit vector pointing from the the curve towards the center of the osculating circle. Then  $\mathbf{N}$  is perpendicular to the unit tangent  $\mathbf{T}$  of the curve, since the osculating circle is also tangential to the curve.  $\mathbf{N}$  is called the **principal normal** to the curve. In general  $\mathbf{T}$  and  $\mathbf{N}$  changes along the curve.  $\mathbf{N}$  is given by

$$(2.3.4) \quad \mathbf{N} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|} = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|}$$

Since  $k = |d\mathbf{T}/ds|$  we can write

$$(2.3.5) \quad \frac{d\mathbf{T}}{ds} = k\mathbf{N}, \quad \mathbf{T} = \frac{d\mathbf{R}}{ds}.$$

**Ex.** Find the curvature and the principal normal of the helix  $\mathbf{R}(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + t \mathbf{k}$

**Sol.**  $\mathbf{R}'(t) = -\sin t \mathbf{i} + \cos t \mathbf{j} + \mathbf{k}$  so  $|\mathbf{R}'(t)| = \sqrt{2}$  and  $\mathbf{T}(t) = \mathbf{R}'(t)/|\mathbf{R}'(t)| = (-\sin t \mathbf{i} + \cos t \mathbf{j} + \mathbf{k})/\sqrt{2}$ .  $\mathbf{T}'(t) = -(\cos t \mathbf{i} + \sin t \mathbf{j})/\sqrt{2}$  and  $|\mathbf{T}'(t)| = 1/\sqrt{2}$ .  $k(t) = |\mathbf{T}'(t)|/|\mathbf{R}'(t)| = 1/2$  and  $\mathbf{N}(t) = \mathbf{T}'(t)/|\mathbf{T}'(t)| = -(\cos t \mathbf{i} + \sin t \mathbf{j})$ .

**Acceleration.** Consider a particle moving along the curve  $\mathbf{R}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ . The velocity is  $\mathbf{v}(t) = \mathbf{R}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$  and the acceleration is

$$(2.3.6) \quad \mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{R}''(t) = x''(t)\mathbf{i} + y''(t)\mathbf{j} + z''(t)\mathbf{k}$$

Since  $|\mathbf{v}| = ds/dt$  we can write

$$(2.3.7) \quad \mathbf{v}(t) = \frac{ds}{dt} \mathbf{T}$$

and hence

$$(2.3.8) \quad \mathbf{a}(t) = \mathbf{v}'(t) = \frac{d^2s}{dt^2} \mathbf{T} + \frac{ds}{dt} \frac{d\mathbf{T}}{ds} = \frac{d^2s}{dt^2} \mathbf{T} + \frac{ds}{dt} \frac{d\mathbf{T}}{ds} \frac{ds}{dt} = \frac{d^2s}{dt^2} \mathbf{T} + \left( \frac{ds}{dt} \right)^2 \kappa \mathbf{N}$$

In other words we have decomposed the acceleration

$$(2.3.9) \quad \mathbf{a} = a_t \mathbf{T} + a_n \mathbf{N}$$

into a component tangential to the path  $a_t = d^2s/dt^2$  and a component  $a_n = k(ds/dt)^2$  perpendicular to the curve. Since  $\mathbf{T}$  and  $\mathbf{N}$  are perpendicular it follows that the magnitude of the total acceleration  $a$  is given by  $a^2 = a_t^2 + a_n^2$ .