Lecture 5: Section 3.1 Scalar Fields. A *scalar field* is a function f that to each point (x, y, y) in a region of space assigns a number $f(x, y, z)$. A physical example of a scalar field is the temperature at each point in space. Mathematically, it is nothing but a scalar valued function of three variables: $f : \mathbf{R}^3 \to \mathbf{R}$.

We now want to measure the rate of change of a vector field f, which will depend on in which direction we go. The **directional derivative** in the direction of a unit vector $\mathbf{u} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ at the point (x_0, y_0, z_0) is given by

$$\frac{df}{ds}(\mathbf{R}_0) = \frac{d}{ds} f(\mathbf{R}(s))\Big|_{s=0}, \qquad \text{where} \quad \mathbf{R}(s) = \mathbf{R}_0 + s\mathbf{u}, \tag{3.1.1}$$

where $\mathbf{R}_0 = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$, or expressed differently

$$\frac{df}{ds}(x_0, y_0, z_0) = \lim_{h\to 0} \frac{f(x_0 + ha, y_0 + hb, z_0 + hc) - f(x_0, y_0, z_0)}{h}. \tag{3.1.2}$$

If we take $a = 1$ and $b = c = 0$ etc. we get the **partial derivatives**:

$$\frac{\partial f}{\partial x}(x, y, z) = \lim_{h\to 0} \frac{f(x + h, y, z) - f(x, y, z)}{h} \tag{3.1.3}$$

$$\frac{\partial f}{\partial y}(x, y, z) = \lim_{h\to 0} \frac{f(x, y + h, z) - f(x, y, z)}{h} \tag{3.1.4}$$

$$\frac{\partial f}{\partial z}(x, y, z) = \lim_{h\to 0} \frac{f(x, y, z + h) - f(x, y, z)}{h} \tag{3.1.5}$$

We define the **gradient** by

$$\nabla f(x, y, z) = \frac{\partial f}{\partial x}(x, y, z)\mathbf{i} + \frac{\partial f}{\partial y}(x, y, z)\mathbf{j} + \frac{\partial f}{\partial z}(x, y, z)\mathbf{k} \tag{3.1.6}$$

The **chain rule** states that, if $\mathbf{R}(t) = x(t)\mathbf{i}+y(t)\mathbf{j}+z(t)\mathbf{k}$ is a space curve depending on the parameter t and $w = f(\mathbf{R}(t))$ then

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial z}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = \nabla f \cdot \frac{d\mathbf{R}}{dt} \tag{3.1.7}$$

The chain rule can be interpreted as saying that the change of the function $\triangle f$ when we change the position $\triangle \mathbf{R} = \triangle x\, \mathbf{i} + \triangle y\, \mathbf{j} + \triangle z\, \mathbf{k}$ is the the sum of the changes of the function when we change the x, y and z directions with $\triangle x$, $\triangle y$ and $\triangle z$ separately. Using the chain rule we can express the directional derivative

$$\frac{df}{ds} = \nabla f \cdot \mathbf{u}. \tag{3.1.8.}$$

Using the formula for the dot product in terms of the angle θ between ∇f and the unit vector $\mathbf{u}$ we see that this is

$$\frac{df}{ds} = |\nabla f| \cos\theta \tag{3.1.9}$$

and it follow that ∇f points in the direction of maximum rate of increase of f and the maximum rate of change is the magnitude of ∇f.

A **level surface** is surface defined by an equation $f(x, y, z) = C$, where f is a scalar field and c is a constant. We note that ∇f is perpendicular to the level surface. In fact if $\mathbf{R}(t)$ is a space curve that lies completely in the surface then since f is constant on the level surface $df(\mathbf{R}(t))/dt = 0$ and by the chain rule

$$\frac{d}{dt} f(\mathbf{R}(t)) = \nabla f \cdot \frac{d\mathbf{R}(t)}{dt} = 0 \tag{3.1.10}$$

i.e. ∇f is perpendicular to the tangent vector $\mathbf{R}'(t)$ to the curve in the surface. Hence ∇f is perpendicular to any tangent vector to any curve in the surface. Therefore ∇f is perpendicular to the surface.

Ex. The level surfaces for the scalar field $f(x, y, z) = x + 2y + 3z$ are planes.

Ex. The level surfaces for the scalar field $f(x, y, z) = x^2 + y^2$ are cylinders.

Ex. The level surfaces for the scalar field $f(x, y, z) = x^2 + y^2 + z^2$ are spheres.

The question now arise if, given a smooth scalar field $f(x, y, z)$, the equation $f(x, y, z) = C$ always defines a surface? The answer is no and a counter example is given by $x^2 + y^2 + z^2 = 0$, which is just satisfied by the point $(0, 0, 0)$. However, if $\nabla f \neq 0$ at a point (x_0, y_0, z_0) then one can show that the equation $f(x, y, z) = f(x_0, y_0, z_0)$ defines a surface for (x, y, z) close to (x_0, y_0, z_0). The proof of this uses the **implicit function theorem** which states that if say $\partial f/\partial z \neq 0$ then one can find a function $z = g(x, y)$ which solves the equation

$$f(x, y, g(x, y)) = f(x_0, y_0, z_0) \tag{3.1.11}$$

for (x, y) close to (x_0, y_0). The idea of the proof is something like that $f(x, y, z)$ increases when we change z in one direction and decreases when we change z in the other direction since $\partial f/\partial z \neq 0$ so any change in f due to a change in x or y can be compensated by a change in z using

$$\triangle f = \frac{\partial f}{\partial x}\triangle x + \frac{\partial f}{\partial y}\triangle y + \frac{\partial f}{\partial z}\triangle z = 0. \tag{3.1.12}$$