Lecture 7: Section 3.4. We define the **curl** of a vector field $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ by

$$\text{(3.4.1)} \qquad \mathbf{curl}\,\mathbf{F} = \Big(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\Big)\mathbf{i} + \Big(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\Big)\mathbf{j} + \Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big)\mathbf{k}$$

One way to remember this formula is that it looks like a cross product:

$$\text{(3.4.2)} \qquad \mathbf{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \begin{vmatrix} \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_2 & F_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial z} \\ F_1 & F_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ F_1 & F_2 \end{vmatrix}\mathbf{k}$$

or with the del or nabla notation:

$$\text{(3.4.3)} \qquad \nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

we symbolically write

$$\text{(3.4.4)} \qquad \mathbf{curl}\,\mathbf{F} = \nabla \times \mathbf{F}$$

In the same way we can symbolically write

$$\text{(3.4.5)} \qquad \text{div}\,\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

and

$$\text{(3.4.6)} \qquad \mathbf{grad}\ f = \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

Ex. 3.4.1 Find $\nabla \times \mathbf{F}$ if $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$.
Sol.

$$\text{(3.4.7)} \quad \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & y & 0 \end{vmatrix} = \begin{vmatrix} \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial z} \\ x & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ x & y \end{vmatrix}\mathbf{k} = \mathbf{0}$$

Ex. 3.4.2 Find $\nabla \times \mathbf{F}$ if $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$.
Sol.
(3.4.8)

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = \begin{vmatrix} \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial z} \\ -y & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ -y & x \end{vmatrix}\mathbf{k} = 2\mathbf{k}$$

The curl tells us how the vector field "swirls" around. From section 3.2 we know that the flow lines for Ex. 3.4.1 are lines going out from the origin where the flow lines for Ex. 3.4.2 are circles around the origin. Ex. 3.4.2 represents the velocity vector field of a body rotating around the z axis at angular velocity 1. In fact, $\mathbf{R}(t) = r\cos t\mathbf{i} + r\sin t\mathbf{j} + c\mathbf{k}$ represents the rotation of a particle at angular velocity 1 and $\mathbf{R}'(t) = \mathbf{F}(\mathbf{R}(t))$, if $\mathbf{F}$ is the vector field in Ex. 3.4.2. Curl is a vector; the magnitude tells us how much it curls and the direction tells us the axis around which it curls. From these examples we might suspect that curl is how much the flow lines curves around, but there is more to it as we shall see.

Ex. 3.4.3 Find $\nabla\times\mathbf{F}$ if $\mathbf{F}=(-y\mathbf{i}+x\mathbf{k})/(x^2+y^2)$. **Sol.**

(3.4.9)
$$\nabla\times\mathbf{F}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ \dfrac{-y}{x^2+y^2} & \dfrac{x}{x^2+y^2} & 0\end{vmatrix}=\Big[\frac{\partial}{\partial x}\Big(\frac{x}{x^2+y^2}\Big)-\frac{\partial}{\partial y}\Big(\frac{-y}{x^2+y^2}\Big)\Big]\mathbf{k}=...=\mathbf{0}$$

It is easy to check that for Ex. 3.4.2 the flow lines are still circles around the z-axis so it appears that our description of the curl as how much the flow lines curves around was not quite sufficient. A more accurate description is that curl is how much the fluid swirls around at a microscopic level at each point. Curl is how much a small paddle wheel "swirls" around its own axis. In Ex. 3.4.2 when the paddle wheel have made a complete rotation around the z-axis it has made a complete rotation around its own axis. However, in Ex. 3.4.3 when the paddle wheel has made a complete rotation around the z-axis it has in fact not rotated around its own axis. In the first case if you are sitting on the paddle wheel facing away from the origin (or z-axis) you are going to face away from the origin the complete rotation around the origin so you will have made a turn as well. However, in the second case you are during the complete rotation around the origin facing in a fixed direction. The explanation for this is that in the second example the velocity vector field gets stronger as we get closer to the origin so the side of the paddle wheel close to the origin will have larger angular velocity. Consider the following example.

Ex. 3.4.4 Find $\nabla\times\mathbf{F}$ if $\mathbf{F}=F_3\mathbf{k}$. **Sol.**

(3.4.9)
$$\nabla\times\mathbf{F}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ 0 & 0 & F_3\end{vmatrix}=\frac{\partial F_3}{\partial y}\mathbf{i}-\frac{\partial F_3}{\partial x}\mathbf{j}$$

Consider in particular the case when F_3 is independent of x but depends on y. The curl can hence be nonvanishing even though in this case the flow lines are straight lines parallel to the z-axis. Going back to the paddle wheel. The magnitude of the curl then is simply the derivative of F_3 in the y direction and that curl is nonvanishing simply means that the velocity vector field is larger on one side of the paddle wheel then on the other.

In the book there is also a discussion of an interpretation of the curl as the "swirl" or rotation of the vector field when we go around a curve divided by the area enclosed by the curve. We will come back to this description later on when we have learned about line integrals along curves. But the basic description is that the component of the curl in a direction of an axis is the integral of the tangential component of the vector field around a small curve in the plane perpendicular to the axis divided by the area of the region enclosed by the curve. If

(3.4.10)
$$\mathbf{curl}\,\mathbf{F}\cdot\mathbf{k}\,\pi r^2=\int_0^{2\pi} f(r\cos\theta,r\sin\theta,c)\,r\,d\theta,\qquad\text{where}\qquad f=\mathbf{F}\cdot\mathbf{T}$$

where $\mathbf{T}$ is the unit tangent to the circle and the component of the curl in the z direction $\mathbf{curl}\,\mathbf{F}\cdot\mathbf{k}$ is evaluated at some point in the area enclosed by the circle. In the book it is further remarked that the same result hold for any small curve in the x-y plane, if πr^2 in the left is replaced by the area of the region enclosed by the curve and if the right hand side is replaced by the integral of the tangential component of the vector field over the curve.