

Lecture 8: Section 4.1: Line Integrals. Let us first recall the definition of arc-length of a curve C parametrized by $\mathbf{R}(t)$, $a \leq t \leq b$. One starts by dividing the curve up into smaller curves, $a = t_0 < t_1 < \dots < t_n = b$, $t_k = a + k\Delta t$, $\Delta t = (b - a)/n$, with endpoints $\mathbf{R}_k = \mathbf{R}(t_k)$. Then

$$(4.1.1) \quad \Delta \mathbf{R}_k = \mathbf{R}(t_k + \Delta t) - \mathbf{R}(t_k) \sim \mathbf{R}'(t_k) \Delta t$$

The arc-length is then is then given by

$$(4.1.2) \quad \int_C ds = \lim_{n \rightarrow \infty} \sum_{k=1}^n |\Delta \mathbf{R}_k| = \lim_{n \rightarrow \infty} \sum_{k=1}^n |\mathbf{R}'(t_k)| \Delta t = \int_a^b |\mathbf{R}'(t)| dt$$

Recall also the arc-length function

$$(4.1.3) \quad s(t) = \int_a^t |\mathbf{R}'(t)| dt, \quad \frac{ds}{dt} = |\mathbf{R}'(t)|$$

The **path integral** of a function f over a curve C is defined by

$$(4.1.4) \quad \int_C f ds = \int_a^b f(\mathbf{R}(t)) |\mathbf{R}'(t)| dt$$

If the curve is in the x - y plan and if the function $f = f(x, y)$ is positive, then this can be interpreted as the area of the surface in space formed by going straight up from the curve to the graph of the function $z = f(x, y)$.

Before we define line integrals let us describe the physical motivation. Suppose that a constant force \mathbf{F} is acting on a particle through a displacement \mathbf{D} . Then the work done on the particle is defined to be

$$(4.1.5) \quad W = \mathbf{F} \cdot \mathbf{D}$$

If instead a vector field $\mathbf{F} = \mathbf{F}(\mathbf{R})$ acts on a particle tracing out a curve $\mathbf{R}(t)$ then the work done by the force moving the particle from \mathbf{R}_{k+1} to \mathbf{R}_k is approximately

$$(4.1.6) \quad \Delta W_k \sim \mathbf{F}(\mathbf{R}_k) \cdot \Delta \mathbf{R}_k \sim \mathbf{F}(\mathbf{R}(t_k)) \cdot \mathbf{R}'(t_k) \Delta t$$

Hence the total work done on the particle is

$$(4.1.7) \quad W = \lim_{n \rightarrow \infty} \sum_{k=1}^n \Delta W_k = \lim_{n \rightarrow \infty} \sum_{k=1}^n \mathbf{F}(\mathbf{R}(t_k)) \cdot \mathbf{R}'(t_k) \Delta t = \int_a^b \mathbf{F}(\mathbf{R}(t)) \cdot \mathbf{R}'(t) dt$$

We therefore define the **line integral** of a vector field \mathbf{F} over the curve C to be

$$(4.1.8) \quad \int_C \mathbf{F} \cdot d\mathbf{R} = \int_a^b \mathbf{F}(\mathbf{R}(t)) \cdot \mathbf{R}'(t) dt = \int_C \mathbf{F} \cdot \mathbf{T} ds$$

where $\mathbf{T} = \mathbf{R}'(t)/|\mathbf{R}'(t)|$ is the unit tangent to the curve and the last integral is the path integral the we previously defined. Since $\mathbf{F} = F_1 \mathbf{i} + F_2 \mathbf{j} + F_3 \mathbf{k}$ and $d\mathbf{R}/dt = \mathbf{i} dx/dt + \mathbf{j} dy/dt + \mathbf{k} dz/dt$ we can write this as

$$(4.1.9) \quad \int_C F_1 dx + F_2 dy + F_3 dz$$

$$= \int_a^b \left(F_1(x(t), y(t), z(t)) \frac{dx}{dt} + F_2(x(t), y(t), z(t)) \frac{dy}{dt} + F_3(x(t), y(t), z(t)) \frac{dz}{dt} \right) dt$$

Ex. 4.1.1 Evaluate $\int_C x^2 dx + xy dy + dz$ where $\mathbf{R}(t) = t\mathbf{i} + t^2\mathbf{j} + \mathbf{k}$, $0 \leq t \leq 1$.

Sol.

$$(4.1.10) \quad \int_0^1 \left(x^2 \frac{dx}{dt} + xy \frac{dy}{dt} + \frac{dz}{dt} \right) dt = \int_0^1 (t^2 + 2t^2) dt = \frac{1}{2}t^3 + \frac{2}{3}t^3 \Big|_0^1 = \frac{11}{15}$$

Ex. 4.1.2 Evaluate $\int_{C_1} \mathbf{F} \cdot d\mathbf{R}$ and $\int_{C_2} \mathbf{F} \cdot d\mathbf{R}$, where $\mathbf{F} = y\mathbf{i} + x\mathbf{j}$, C_1 is the curve

$\mathbf{R}_1(t) = (1-t)\mathbf{i}$, $0 \leq t \leq 1$ and $\mathbf{R}_1(t) = (t-1)\mathbf{i} + (t-1)\mathbf{j}$, when $1 \leq t \leq 1 + 1/\sqrt{2}$ and C_2 is the curve $\mathbf{R}_2(t) = \cos t\mathbf{i} + \sin t\mathbf{j}$, $0 \leq t \leq \pi/4$. Conclusion?

Sol.: We divide C_1 up into two parts. When $0 \leq t \leq 1$ then $x = (1-t)$ and $y = 0$ so $dx/dt = -1$ and $dy/dt = 0$ and when $1 \leq t \leq 1 + 1/\sqrt{2}$ we have $x = (t-1)$, $y = (t-1)$ and $dx/dt = dy/dt = 1$ so

$$(4.1.11) \quad \int_{C_1} \mathbf{F} \cdot d\mathbf{R} = \int_0^1 \left(y \frac{dx}{dt} + x \frac{dy}{dt} \right) dt + \int_1^{1+1/\sqrt{2}} \left(y \frac{dx}{dt} + x \frac{dy}{dt} \right) dt$$

$$\int_0^1 0 dt + \int_1^{1+1/\sqrt{2}} ((t-1)1 + (t-1)) dt = (t-1)^2 \Big|_1^{1+1/\sqrt{2}} = \frac{1}{2}$$

On C_2 we have $x = \cos t$, $y = \sin t$, $dx/dt = -\sin t$, $dy/dt = \cos t$ so

$$(4.1.12) \quad \int_{C_2} \mathbf{F} \cdot d\mathbf{R} = \int_0^{\pi/4} \left(y \frac{dx}{dt} + x \frac{dy}{dt} \right) dt$$

$$\int_0^{\pi/4} (-\sin^2 t + \cos^2 t) dt = \int_0^{\pi/4} \cos(2t) dt = \frac{\sin(2t)}{2} \Big|_0^{\pi/4} = \frac{1}{2}$$

Both line integrals go from $(0, 1)$ to $(1/\sqrt{2}, 1/\sqrt{2})$ over different paths. In this case the value of the line integral is independent of the path. Vector fields for which this is true are called conservative and we will study them in section 4.3.

Section 4.2: Domains: Simply connected domains. We have already encountered functions and vector fields that are not defined everywhere, e.g. $f(x, y, z) = 1/(x^2 + y^2 + z^2)$ is not defined at the origin and $\mathbf{F} = (-y\mathbf{i} + x\mathbf{j})/(x^2 + y^2)$ is not defined along the z -axis. Therefore it is natural to talk about these functions being defined and continuous in the region obtained by removing the origin respectively the z -axis. We will say that a region is **open** if, around any point in the region there is a small ball which is completely contained in the region. In some sense, this says that the region is "thick", i.e. there is a some space around each point in each direction. E.g. a surface in space is not open since if you go in the direction of the normal you immediately leave the surface. We define the **boundary** of a region to be the points such that every small ball around them contains points both inside and outside the region. We say that an open region is **connected** if given any two points in the region there is a continuous curve in the region connecting them. A region that is open and connected is called a **domain**. A connected region such that every closed curve in it can be continuously shrunk to a point without leaving the domain, is called **simply connected**.

Ex. Space with the origin removed is connected and simply connected. Space with the z -axis removed is connected but not simply connected. The plane with the origin removed is however connected but not simply connected.

Ex. A plane in space or a line in the plane are closed sets. A ball without the boundary is an open set.