

Solutions to MATH 20E FINAL EXAM Fall 98, Lindblad, Problems 1-5 only.

1. (a) $\nabla f = 6xye^{x^2y-2}\mathbf{i} + 3x^2e^{x^2y-2}\mathbf{j}$ so it increases most in the direction $12\mathbf{i} + 3\mathbf{j}$.
 b) $z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$ so $z - 3 = 12(x - 1) + 3(y - 2)$.

2. a) $\nabla \times \mathbf{F} = 0$.

$$b) \int_C \mathbf{F} \cdot d\mathbf{R} = \int_C \frac{xdx + ydy}{x^2 + y^2} = \int_0^{2\pi} \frac{r \cos t r(-\sin t) + r \sin t r \cos t}{r^2} dt = 0.$$

- c) $\mathbf{F} = \nabla\phi$, where $\phi = \ln \sqrt{x^2 + y^2}$.

- d) The flow lines are lines from the origin: $\frac{dx}{x/(x^2 + y^2)} = \frac{dy}{y/(x^2 + y^2)}$ so $\frac{dx}{x} = \frac{dy}{y}$ and hence $\ln|x| + C = \ln|y|$ which is equivalent to $|y| = e^C|x|$.

3. a) $\nabla \times \mathbf{F} = 0$.

$$b) \int_C \mathbf{F} \cdot d\mathbf{R} = \int_C \frac{ydx - xdy}{x^2 + y^2} = \int_0^{2\pi} \frac{r \sin t r(-\sin t) - r \cos t r \cos t}{r^2} dt = -\int_0^{2\pi} dt = -2\pi$$

- c) \mathbf{F} is not conservative since the integral around the closed curve in (b) is $\neq 0$.

- d) The flow lines are circles centered at the origin: $\frac{dx}{y/(x^2 + y^2)} = \frac{dy}{-x/(x^2 + y^2)}$ so $-x dx = y dy$ and hence $y^2 = -x^2 + C$ which is equivalent to $x^2 + y^2 = C$.

4. a) $F(0, 2) = (2, 0)$, $F(0, 4) = (0, 4)$, $F(\pi/2, 2) = (0, 2)$ and $F(\pi/2, 4) = (0, 4)$.

- b) $\frac{\partial(x, y)}{\partial(u, v)} = 2v$.

$$c) \text{Area} = \iint_S dx dy = \iint_R \frac{\partial(x, y)}{\partial(u, v)} du dv = \int_0^{\pi/2} \int_2^4 2v du dv = \dots = \frac{\pi}{2} (4^2 - 2^2)$$

5. a) $0 \leq z \leq 1$ implies that $3 \leq x^2 + y^2 \leq 4$ so we can write S in the form of a graph $z - 1 = g(x, y) = -\sqrt{4 - x^2 - y^2}$ where $(x, y) \in D = \{(x, y); 3 \leq x^2 + y^2 \leq 4\}$.

$$dS = dx dy / |\mathbf{n} \cdot \mathbf{k}| = \sqrt{1 + g_x^2 + g_y^2} dx dy = \dots = 2 dx dy / \sqrt{4 - x^2 - y^2} \text{ so}$$

$$\text{Area}(S) = \iint_S dS = \iint_D \frac{dx dy}{\sqrt{4 - x^2 - y^2}} = \int_{\sqrt{3}}^2 \int_0^{2\pi} \frac{r dr d\theta}{\sqrt{4 - r^2}} = -2\pi \sqrt{4 - r^2} \Big|_{\sqrt{3}}^2 = 2\pi$$

$$\begin{aligned} \iiint_B x^2 dx dy dz &= \int_0^2 \int_0^{2\pi} \int_0^\pi r^2 \sin^2 \phi \cos^2 \theta r^2 \sin \phi d\phi d\theta dr = \int_0^2 \int_0^{2\pi} -\frac{\cos^3 \phi}{3} \Big|_0^\pi \cos^2 \theta r^4 d\theta dr \\ &= \frac{2}{3} \frac{r^5}{5} \Big|_0^2 \int_0^{2\pi} \cos^2 \theta d\theta = \frac{2}{3} \frac{2^5}{5} \pi \end{aligned}$$

NOTE: THERE ARE NO SOLUTIONS TO PROBLEMS 6-9!