Solutions to Math 20E Final Winter 99, Lindblad.

1. Adding the two equations together gives $3x - y + 2 = 0$ so $y = 3x + 2$. Inserting this in the first equation gives $z = 3y - 2x + 1 = 3(3x+2) - 2x + 1 = 7x + 7$. Hence the equation of the plane is $(x, y, z) = (t, 3t+2, 7t+7) = (1, 3, 7)t + (0, 2, 7)$.

2. (a). $\mathbf{F}(x, y, z) = \nabla\phi = y^2\mathbf{i} + 2xy\mathbf{j}$.

(b). Along the curve we have $\dfrac{dx}{y^2} = \dfrac{dy}{2xy} = \dfrac{dz}{0}$ so $ydy = 2xdx$ and $dz = 0$. Hence $y^2 = 2x^2 + C$ so $y = \pm\sqrt{2x^2 + C_1}$ and $z = C_2$ Since the curve starts at $(1, -1, 1)$ we must have $(x, y, z) = (t, -\sqrt{2t^2 - 1}, 1)$

3. (a) $\phi_x = ze^{x-y}$, $\phi_y = -ze^{x-y}$ and $\phi_z = e^{x-y}$. Integrating these equations give $\phi = ze^{x-y} + h_1(y, z)$, $\phi_y = ze^{x-y} + h_2(x, z)$ and $\phi = ze^{x-y} + h_3(x, y)$ which can all be satisfied if we pick $h_1 = h_2 = h_3 = 0$ in wchich case $\phi = ze^{x-y}$.

(b) The end point of the curve is $(1, 0, 0)$ and the initial point is $(0, 0, 1)$ so $\int_C \mathbf{F}\cdot\mathbf{R} = \phi(1, 0, 0) - \phi(0, 0, 1) = -1$.

(c) If $\mathbf{G}$ denotes the vector field then $\nabla \times \mathbf{G} = 0$ and G is continuously differentiable in the region obtained by removing the origin from space. Since this region is simply connected it follows that there exists a potential so $\mathbf{G}$ is conservative. Solving the equations we see that indeed $\psi = \ln\sqrt{x^2 + y^2 + z^2}$ is a potential so it is conservative.

4. (a) See handout.

(b) $R = \{(u, v);\ 1 \le v^2 \le 2,\ 2\sinh u \le \cosh u \le 3\sinh u\}$. The last two inequalities means that $e^u - e^{-u} = (e^u + e^{-u})/2 \le 3(e^u - e^{-u})/2$ and multiplying by e^u gives $e^{2u} - 1 \le e^{2u}/2 + 1/2 \le 3e^{2u}/2 - 3/2$ or $e^{2u}/2 \le 3/2$ respectively $2 \le e^{2u}$ so $2 \le e^{2u} \le 3$. or $\ln 2 \le 2u \le \ln 3$. We get $R = \{(u, v);\ \ln 2/2 \le u \le \ln 3/2,\ 1 \le v \le \sqrt{2}\}$.

(c) $\dfrac{\partial(x, y)}{\partial(u, v)} = -v$.

(d) $$\iint_R \frac{dxdy}{(1 + x^2 - y^2)^2} = \iint_R \frac{vdudv}{(1 + v^2)^2} = \int_a^b \frac{-1/2}{1 + v^2}\bigg|_c^d du = \frac{a - b}{2}\left(\frac{1}{1 + d^2} - \frac{1}{1 + c^2}\right)$$

5. (a) See book. (b) $A = \text{Area}(D) = \frac{1}{2}\int_C xdy - ydx$

$$(c) \quad A = \frac{1}{2}\int_C xdy - ydx = \frac{1}{2}\int_{-\sqrt{3}}^{\sqrt{3}} t^2(t^2-1) - \Big(\frac{t^3}{3} - t\Big)3t\,dt = \int_{-\sqrt{3}}^{\sqrt{3}} t^2\,dt$$

$$= \frac{t^3}{3}\Big|_{-\sqrt{3}}^{\sqrt{3}} = 2\sqrt{3}$$

$$(d) \quad L = \int_C ds = \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{(dx/dt)^2 + (dy/dt)^2}dt = \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{(2t)^2 + (t^2-1)^2}dt$$

$$= \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{(t^2+1)^2}dt = \frac{t^3}{3} + t\Big|_{-\sqrt{3}}^{\sqrt{3}} = 4\sqrt{3}$$

6. (a) See book. (b) The exterior unit normal is $\mathbf{n} = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})/a$.

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_S \frac{x+y+z^4}{a}dS = \int_0^{\pi}\int_0^{2\pi} \frac{a\sin\phi\cos\theta + a\sin\phi\sin\theta + a^4\cos^4\phi}{a}\,a^2\sin\phi\,d\theta d\phi$$

$$= \int_0^{\pi} 2\pi a^5\cos^4\phi\sin\phi\,d\phi = 2\pi a^5\frac{-\cos^5\phi}{5}\Big|_0^{\pi} = \frac{4\pi a^5}{5}$$

(c) Let B be the ball of radius a.

$$(c) \iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_B \text{div}\mathbf{F}dV = \iiint_B 3z^2dV = \int_0^{\pi}\int_0^{2\pi}\int_0^{a} 3r^2\cos^2\phi\, r^2\sin\phi\,drd\theta d\phi$$

$$= \int_0^{\pi}\int_0^{2\pi} \frac{3r^5}{5}\Big|_0^{a}\cos^2\phi\sin\phi d\theta d\phi = \frac{3a^5}{5}2\pi\frac{-\cos^3\phi}{3}\Big|_0^{\pi} = \frac{3a^5}{5}2\pi\frac{2}{3}$$

7. (a) See book. (b) C is parametrized by $x = \cos t$, $y = \sin t$, $z = 1$, $0 \le t \le 2\pi$.

$$\int_C \mathbf{F}\cdot d\mathbf{R} = \int_C ydx - xdy + zdz = \int_0^{2\pi}(-\sin^2 t - \cos^2 t)dt = -2\pi$$

$$(c) \quad \int_C \mathbf{F}\cdot d\mathbf{R} = \iint_S \nabla\times\mathbf{F}\cdot\mathbf{n}\,dS = \iint_D -2\mathbf{k}\cdot n\frac{dxdy}{\mathbf{n}\cdot\mathbf{k}} = -2\iint_D dxdy = -2\pi$$

8. See solution to problem 4 on midterm 2 Spring 2000.