Formulas for Midterm 1.

Curve C: $\mathbf{c}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, $a \le t \le b$.

Tangent vector: $\mathbf{c}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$.

Arc length: $L = \displaystyle\int_a^b \|\mathbf{c}'(t)\|\,dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt$

Gradient: $\nabla f(x,y,z) = \mathbf{grad}\, f(x,y,z) = \dfrac{\partial f}{\partial x}(x,y,z)\mathbf{i} + \dfrac{\partial f}{\partial y}(x,y,z)\mathbf{j} + \dfrac{\partial f}{\partial z}(x,y,z)\mathbf{k}$

Chain rule case 1: If $h(t) = f(\mathbf{c}(t))$ then $\dfrac{dh}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial z}\dfrac{dy}{dt} + \dfrac{\partial f}{\partial z}\dfrac{dz}{dt} = \nabla f \cdot \dfrac{d\mathbf{c}}{dt}$

Directional derivative: $\dfrac{d}{dt} f\big(\mathbf{c}(t)\big)\Big|_{t=0} = \nabla f \cdot \mathbf{u}$, if $\mathbf{c}(t) = \mathbf{x}_0 + t\mathbf{u}$, $|\mathbf{u}| = 1$.

Linear approximation $f(\mathbf{x}_0 + \mathbf{h}) \sim f(\mathbf{x}_0) + \displaystyle\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\mathbf{x}_0) h_i = f(\mathbf{x}_0) + \mathbf{D}f(\mathbf{x}_0)\mathbf{h}$,

where the derivative matrix $\mathbf{D}f = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \dfrac{\partial f_m}{\partial x_1} & \cdots & \dfrac{\partial f_m}{\partial x_n} \end{bmatrix}$ and $\mathbf{h} = (h_1, \ldots, h_n) = \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix}$.

2nd order Taylor formula $f(\mathbf{x}_0 + \mathbf{h}) \sim f(\mathbf{x}_0) + \displaystyle\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\mathbf{x}_0) h_i + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}_0) h_i h_j$

Chain rule: Suppose $g : \mathbf{R}^n \to \mathbf{R}^m$, $f : \mathbf{R}^m \to \mathbf{R}^p$ and let $h : f \circ g : \mathbf{R}^n \to \mathbf{R}^p$ be $h(x) = f \circ g(x) = f(g(x))$. Then $\mathbf{D}h(\mathbf{x}_0) = \mathbf{D}f(\mathbf{y}_0)\,\mathbf{D}g(x_0)$, where $\mathbf{y}_0 = g(\mathbf{x}_0)$.

Vector Field: $\mathbf{F}(x,y,z) = F_1(x,y,z)\mathbf{i} + F_2(x,y,z)\mathbf{j} + F_3(x,y,z)\mathbf{k}$.

Flow line: $\dfrac{d\mathbf{c}(t)}{dt} = \beta\,\mathbf{F}\big(\mathbf{c}(t)\big), \quad \beta \ge 0$ i.e. $\begin{cases} x' &= \beta F_1 \\ y' &= \beta F_2 \\ z' &= \beta F_3 \end{cases}$ or $\dfrac{dx}{F_1} = \dfrac{dy}{F_2} = \dfrac{dz}{F_3}$.

Divergence: $\nabla \cdot \mathbf{F} = \operatorname{div} \mathbf{F} = \dfrac{\partial F_1}{\partial x} + \dfrac{\partial F_2}{\partial y} + \dfrac{\partial F_3}{\partial z}$.

Curl: $\nabla \times \mathbf{F} = \mathbf{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \begin{vmatrix} \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_2 & F_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial z} \\ F_1 & F_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ F_1 & F_2 \end{vmatrix}\mathbf{k}$,

where $\begin{vmatrix} \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_2 & F_3 \end{vmatrix} = \dfrac{\partial F_3}{\partial y} - \dfrac{\partial F_2}{\partial z}$ etc.