20F Homework 1.
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1.1.16: Describe the line passing through $(-5, 0, 4)$ and $(6, -3, 2)$.

This line is in the direction $v = (6, -3, 2) - (-5, 0, 4) = (11, -3, -2)$, and $(-5, 0, 4)$ is a point on the line. Thus the set $\{ (x, y, z) \in \mathbb{R}^3 :$
$(x, y, z) = (-5, 0, 4) + t(11, -3, -2),\ t \in \mathbb{R} \}$
describes this line.

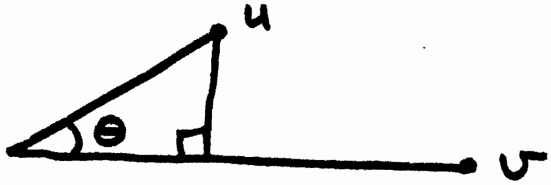
1.2.8: Compute $\|u\|$, $\|v\|$ and $u \cdot v$, where $u = (5, -1, 2)$ and $v = (1, 1, -1)$.

$$\|u\| = \sqrt{(5)^2 + (-1)^2 + (2)^2} = \sqrt{25 + 1 + 4} = \sqrt{30}$$

$$\|v\| = \sqrt{1^2 + 1^2 + (-1)^2} = \sqrt{3}$$

$$u \cdot v = 5 \cdot 1 + (-1) \cdot 1 + (2) \cdot (-1) = 5 - 1 - 2 = 2.$$

1.2.14: Find the projection of $u = (-1, 1, 1)$ onto $v = (2, 1, -3)$.



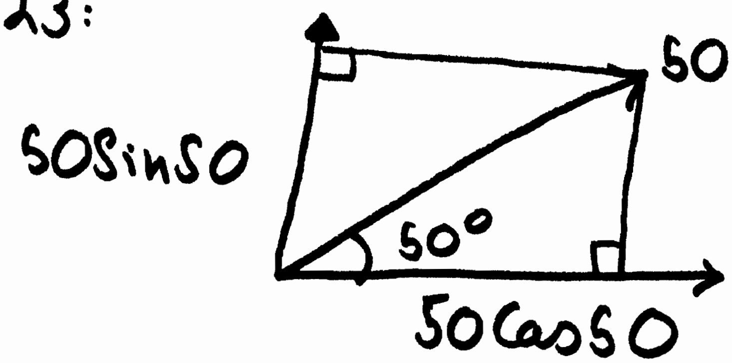
The length of the projection of u onto v is $\|u\| \cos\theta$, and the projection points in the direction of $\frac{v}{\|v\|}$. Thus the projection is

$$\|u\| \cos\theta \left(\frac{v}{\|v\|}\right) = \frac{u \cdot v}{\|v\|^2} v$$

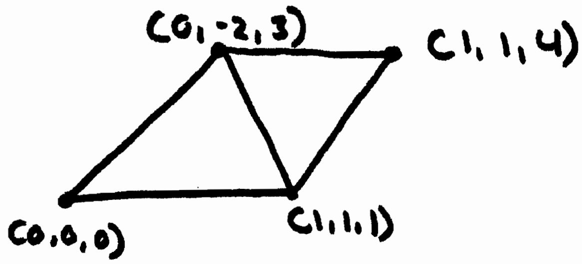
$$= \left[\frac{((-1)\cdot 2 + 1\cdot 1 + 1\cdot(-3))}{2^2 + 1^2 + (-3)^2}\right] v$$

$$= \left[\frac{-2+1-3}{14}\right] v = \left(\frac{-4}{14}\right) v.$$

1.2.23:



1.3.6: Find the area of a triangle with verticies $(0,0,0)$, $(1,1,1)$ and $(0,-2,3)$.



the area of the parallelogram is

$$\|(1,1,1) \times (0,-2,3)\| = \left\| \begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 0 & -2 & 3 \end{matrix} \right\| = \| \hat{i}[3+2] - \hat{j}[3] + \hat{k}[-2] \|$$

$$= \|(5,-3,-2)\| = \sqrt{5^2+(-3)^2+(-2)^2} = \sqrt{25+9+4} = \sqrt{38}.$$

The area of the triangle is half of this: $\frac{\sqrt{38}}{2}$.

1.3.7: What is the volume of the parallelepiped with sides $(2,1,-1)$, $(5,0,-3)$, and $(1,-2,1)$?

This is given by the absolute value of $(2,1,-1)\cdot[(5,0,-3)\times(1,-2,1)]$.

$$(5,0,-3)\times(1,-2,1) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 5 & 0 & -3 \\ 1 & -2 & 1 \end{vmatrix}$$

$$= \hat{\imath}[-6] - \hat{\jmath}[5+3] + \hat{k}[-10]$$

$$= (-6,-8,-10)$$

thus

$$|(2,1,-1)\cdot(-6,-8,-10)| = |-12-8+10|$$

$$= |-10|$$

$$= 10.$$

1.3.12: Describe all unit vectors in $\mathbb{R}^3$ which are orthogonal to $(2,-4,3)$, and $(-4,8,-6)$.

In $\mathbb{R}^3$ there are only two such vectors. To find a vector which is orthogonal to the two vectors we compute the cross product:

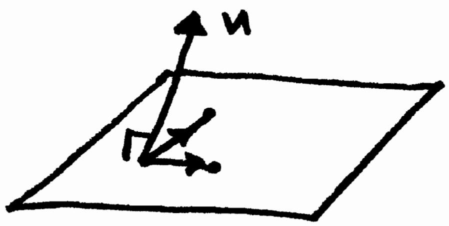
$$(2,-4,3)\times(-4,8,-6) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -4 & 3 \\ -4 & 8 & -6 \end{vmatrix}$$

$$= \hat{i}\,[24-24] - \hat{j}\,[-12+12] + \hat{k}\,[16-16]$$

$$= 0$$

In this case it turns out that the vectors are parallel. Thus there are infinitely many vectors which are orthogonal. They look like (x,y,z), with the condition that $(x,y,z)\cdot(2,-4,3) = 2x-4y+3z = 0$ and $\|(x,y,z)\| = 1$.

1.3.16a: Find the equation of the plane which passes through $(0,0,0)$, $(2,0,-1)$, and $(0,4,-3)$.



First we find the normal n to the plane. The vectors $(0,4,-3) - (0,0,0)$ and $(2,0,-1) - (0,0,0)$ are in the plane. Thus the vector

$$(0,4,-3) \times (2,0,-1) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 4 & -3 \\ 2 & 0 & -1 \end{vmatrix}$$

$$= \hat{i}[-4] - \hat{j}[6] + \hat{k}[-8]$$

$$= (-4, -6, -8)$$

is normal. Thus all vectors in the plane (x,y,z) must satisfy

$$0 = (x, y, z) \cdot (-4, -6, -8)$$
$$= -4x - 6y - 8z,$$

thus the plane is given by $0 = 2x + 3y + 4z$.

1.5.7: Let $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 3 & 2 \\ 3 & 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 0 & 2 \\ -1 & 1 & -1 \\ 1 & 4 & 3 \end{bmatrix}$.

Compute $A \cdot B$, $\mathrm{Det}(A)$, $\mathrm{Det}(B)$ and $\mathrm{Det}(A+B)$.

$$A \cdot B = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 3 & 2 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 & 2 \\ -1 & 1 & -1 \\ 1 & 4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -2+1 & -1 & 2+1 \\ -3+2 & 3+8 & -3+6 \\ -6-1+1 & 1+4 & 6-1+3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -1 & 3 \\ -1 & 11 & -3 \\ -6 & 5 & 8 \end{bmatrix}.$$

$$\mathrm{Det}(A) = 1 \cdot (3-2) + 3 \cdot (-2)$$

$$= 1 - 6 = -5.$$

The other computations are similar.

1.5.18: Show that if $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible, the solution of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e \\ f \end{pmatrix}$ is $\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} e \\ f \end{pmatrix}$.

By exercise 17, we know that the inverse of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Thus $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e \\ f \end{pmatrix}$, gives, when multiplied on the left by the inverse of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} e \\ f \end{pmatrix}.$$

Review. 29: Let A, B be $n \times n$ matrices. Let $x \in \mathbb{R}^n$. Show that $(AB)x = A(Bx)$.

Let $C = \begin{bmatrix} x & 0 & \cdots & 0 \\ & \vdots & & \vdots \\ & 0 & \cdots & 0 \end{bmatrix}$ be an $n \times n$ matrix. Then since matrix multiplication is associative $(AB)C = A(BC)$. Now $(AB)C = \begin{bmatrix} (AB)x & 0 & \cdots & 0 \\ & \vdots & & \vdots \\ & 0 & \cdots & 0 \end{bmatrix}$

And $A(BC) = A\begin{bmatrix} Bx & \begin{matrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{matrix} \end{bmatrix}$.

Thus $(AB)x = A(Bx)$.

The above implies that the composition of the maps $x \longmapsto Ax$ and $y \longmapsto By$ is given by $z \longmapsto (AB)z$.

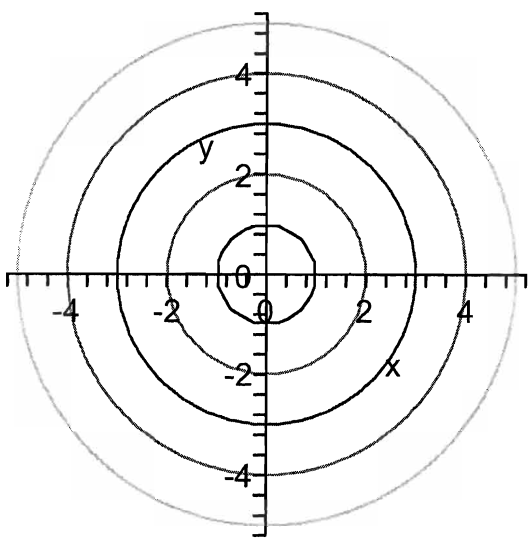
Section 1.4, Problem 4:
(a) Describe the surfaces r = constant, θ = constant, and z = constant in cylindrical coordinates.
(b) Describe the surfaces ρ = constant, θ = constant, and ϕ = constant in spherical coordinates.

Solution. In cylindrical coordinates, the surface r = constant is a cylinder of radius r; the surface θ = constant is a plane through the origin that subtends an angle θ against the plane $y = 0$; the surface z = constant is a horizontal plane.

In spherical coordinates, the surface ρ = constant is a sphere of radius ρ; the surface θ = constant is a half-plane terminating at the z-axis and subtending an angle of θ against the plane $y = 0$; the surface ϕ = constant is a cone based at the origin created from all the rays emanating from the origin at an angle of ϕ from the positive z-axis.

Section 2.1, Problem 6: Sketch the level curves of $(x^2 + y^2)^{1/2}$ for $c = 0, 1, 2, 3, 4, 5$.

Solution.



2.1.11

Describe the level surfaces and a section of the function $f(x,y,z) = -x^2 - y^2 - z^2$

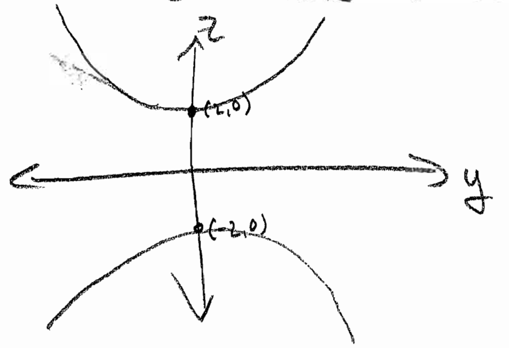
If $c > 0$ then $c = -x^2 - y^2 - z^2$ has no solution.

$c = 0$ then $0 = x^2 + y^2 + z^2 \Rightarrow (x,y,z) = (0,0,0)$ and the level surface corresponding to the function value 0 is the origin.

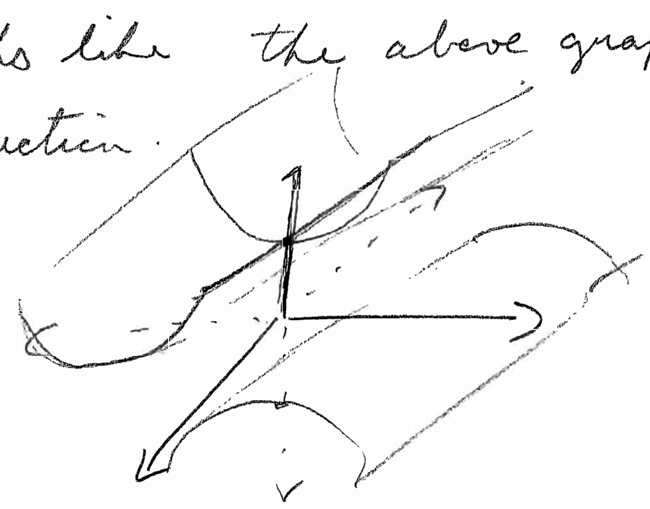
$c < 0$ then $-c = x^2 + y^2 + z^2$ gives a sphere of radius $\sqrt{-c}$.

2.1.21

The surface $z^2 = y^2 + 4$ in $\mathbb{R}^3$ has a cross-section for any $x = c$ of



So the surface looks like the above graph extended in the x-direction.



2.1.28. $y^2 = x^2 + z^2$. This one should just recognize as being a cone.