Hw 3 Fall 2009

Section 4.2, Problem 2: Find the arc length of the curve $(1, 3t^2, t^3)$ over the interval $0 \le t \le 1$.

Solution. The arc length is

$$\begin{aligned} \int_0^1 \|(0, 6t, 3t^2)\| dt &= \int_0^1 \sqrt{36t^2 + 9t^4}\, dt \\ &= \int_0^1 3t\sqrt{4+t^2}\, dt \\ &= (4+t^2)^{3/2}\Big|_0^1 \\ &= 5\sqrt{5} - 8. \end{aligned}$$

Section 4.2, Problem 10: The arc length function $s(t)$ for a given path $\mathbf{c}(t)$, defined by $s(t) = \int_a^t \|\mathbf{c}'(\tau)\|\, d\tau$, represents the distance a particle traversing the trajectory of $\mathbf{c}$ will have travelled by time t if it starts out at a; that is, it gives the length of $\mathbf{c}$ between $\mathbf{c}(a)$ and $\mathbf{c}(t)$. Find the arc length functions for the curves $\alpha(t) = (\cosh t, \sinh t, t)$ and $\beta(t) = (\cos t, \sin t, t)$, with $a = 0$.

Solution. For $\alpha(t)$, we get

$$\begin{aligned} s_\alpha(t) &= \int_0^t \sqrt{\sinh^2\tau + \cosh^2\tau + 1}\, d\tau \\ &= \int_0^t \sqrt{\frac{2e^{2\tau} + 2e^{-2\tau}}{4} + 1}\, d\tau \\ &= \sqrt{2}\int_0^t \frac{e^\tau + e^{-\tau}}{2}\, d\tau \\ &= \sqrt{2}\sinh t. \end{aligned}$$

For $\beta(t)$ we get

$$\begin{aligned} s_\beta(t) &= \int_0^t \sqrt{(-\sin\tau)^2 + \cos^2\tau + 1^2}\, d\tau \\ &= \int_0^t \sqrt{2}\, d\tau \\ &= \sqrt{2}t. \end{aligned}$$

Section 4.2, Problem 11: Let $\mathbf{c}(t)$ be a given path, $a \le t \le b$. Let $s = \alpha(t)$ be a new variable, where α is a strictly increasing C^1 function given on $[a, b]$. For each s in $[\alpha(a), \alpha(b)]$ there is a unique t with $\alpha(t) = s$. Define a function $\mathbf{d} : [\alpha(a), \alpha(b)] \to \mathbb{R}^3$ by $\mathbf{d}(s) = \mathbf{c}(t)$.

(a) Argue that the images of the curves $\mathbf{c}$ and $\mathbf{d}$ are the same.
(b) Show that $\mathbf{c}$ and $\mathbf{d}$ have the same arc length.
(c) Let $s = \alpha(t) = \int_a^t \|\mathbf{c}'(\tau)\| d\tau$. Define $\mathbf{d}$ as above by $\mathbf{d}(s) = \mathbf{c}(t)$. Show that

$$\left\| \frac{d}{ds}\mathbf{d}(s) \right\| = 1.$$

Solution.

(a) For every point on $\mathbf{c}(t)$, we have $\mathbf{d}(\alpha(t)) = \mathbf{c}(t)$, so every point on $\mathbf{c}$ is a point on $\mathbf{d}$. For every point on $\mathbf{d}(s)$, since α is continuous and ranges from $\alpha(a)$ to $\alpha(b)$, there is some $t \in [a,b]$ such that $\alpha(t) = s$, and so $\mathbf{c}(t) = \mathbf{d}(s)$. So every point on **d** is a point on **c**, which means that the images of the curves are the same.

(b) The arc length of **d** is

$$s_{\mathbf{d}} = \int_{\alpha(a)}^{\alpha(b)} \|\mathbf{d}'(s)\| \, ds$$

Making the substitution $s = \alpha(t)$, our bounds change from $[\alpha(a), \alpha(b)]$ to $[a,b]$, and we have $ds = \alpha'(t)\, dt$. We then have:

$$\begin{aligned} s_{\mathbf{d}} &= \int_a^b \|\mathbf{d}'(\alpha(t))\| \cdot \alpha'(t) \, dt \\ &= \int_a^b \|\mathbf{d}'(\alpha(t)) \cdot \alpha'(t)\| \, dt; \end{aligned}$$

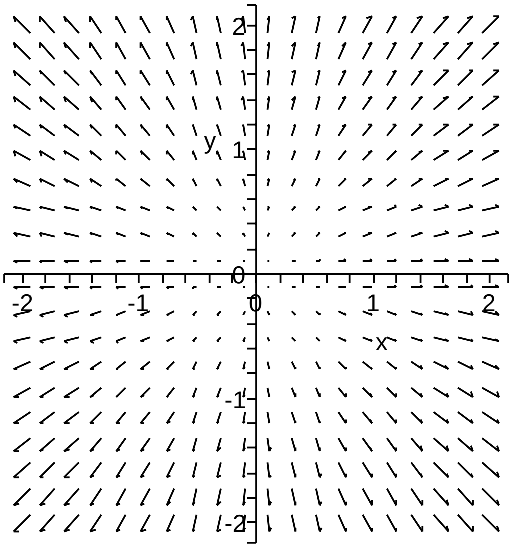
since α is increasing we can pull it inside the absolute value bar. But that last integral is the arc length formula for $\mathbf{c}(t)$; therefore, the two curves have equal arc length.

(c) First, we use the fundamental theorem of calculus to get $ds/dt = \alpha'(t) = \|\mathbf{c}'(t)\|$. We use that to get:

$$\left\| \frac{d}{ds}\mathbf{d}(s) \right\| = \left\| \frac{d}{ds}\mathbf{c}(t) \right\| = \left\| \mathbf{c}'(t) \cdot \frac{dt}{ds} \right\| = \left\| \mathbf{c}'(t) \cdot \frac{1}{\mathbf{c}'(t)} \right\| = 1.$$

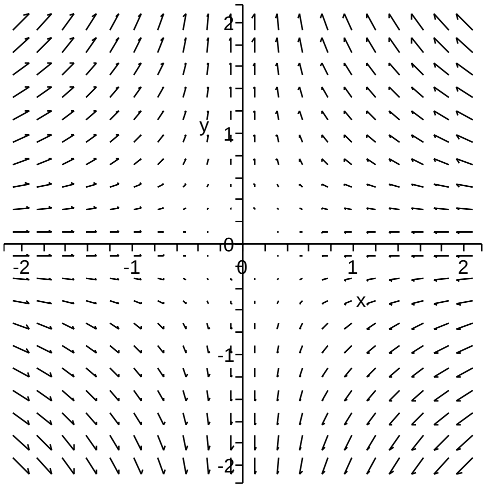
Section 4.3, Problem 3: Sketch the vector field $\mathbf{F}(x, y) = (x, y)$.

Solution.



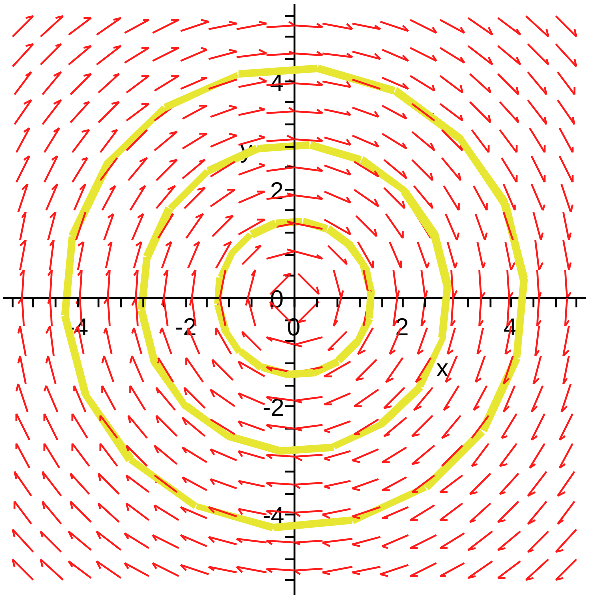
Section 4.3, Problem 4: Sketch the vector field $\mathbf{F}(x, y) = (-x, y)$.

Solution.



Section 4.3, Problem 9: Sketch a few flow lines of the vector field $\mathbf{F}(x,y) = (y,-x)$.

Solution.

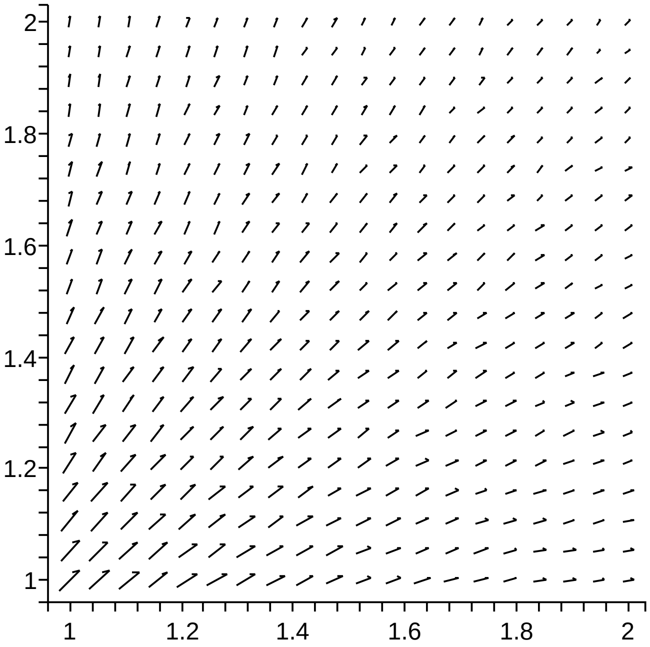


Section 4.3, Problem 14: Show that $\mathbf{c}(t) = (t^2, 2t-1, \sqrt{t})$, $t > 0$ is a flow line of the velocity vector field $\mathbf{F}(x,y,z) = (y+1, 2, 1/2z)$.

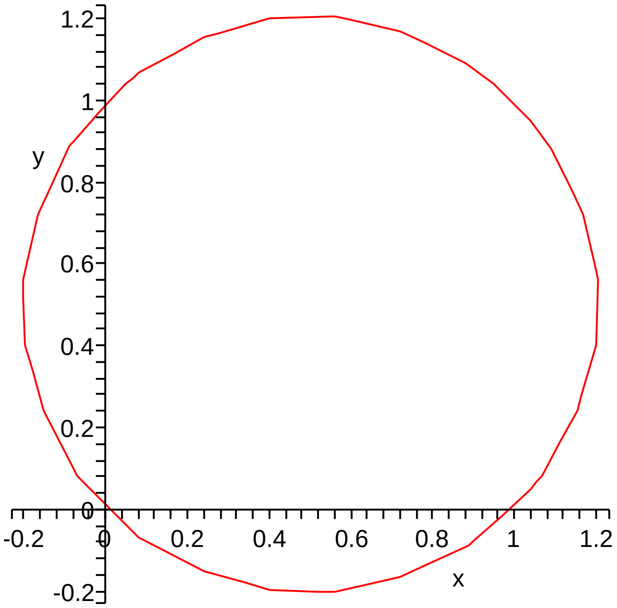
Solution. We have $\mathbf{c}'(t) = (2t, 2, 1/2\sqrt{t})$, and $\mathbf{F}(\mathbf{c}(t)) = (2t, 2, 1/2\sqrt{t})$. Since they are equal, $\mathbf{c}$ is a flow line for $\mathbf{F}$.

Section 4.3, Problem 20: Sketch the gradient field $-\nabla V$ for $V(x,y) = (x+y)/(x^2+y^2)$ and the equipotential surface $V = 1$.

Solution. We have $-\nabla V = (\frac{x^2+2xy-y^2}{(x^2+y^2)^2}, \frac{-x^2+2xy+y^2}{(x^2+y^2)^2})$. The gradient field looks like:



The equipotential surface $V = 1$ is a circle centered in the first quadrant:



Section 4.4, Problem 4: Find the divergence of $\mathbf{V}(x, y, z) = (x^2, (x+y)^2, (x+y+z)^2)$.

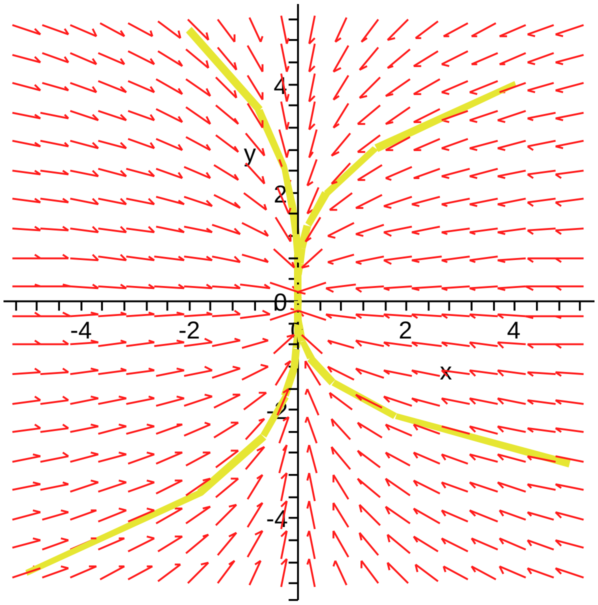
Solution. The divergence is $2x + 2(x+y) + 2(x+y+z) = 6x + 4y + 2z$.

Section 4.4, Problem 5: Figure 4.4.11 shows some flow lines and moving regions for a fluid moving in the velocity field **V**. Where is div $\mathbf{V} > 0$, and where is div $\mathbf{V} < 0$?

Solution. In the first and third quadrants, the lines and regions are pushing away from the origin, so the divergence is positive. In the second and fourth quadrants, the lines and regions are pushing towards the origin, so the divergence there is negative.

Section 4.4, Problem 8: Sketch a few flow lines for $\mathbf{F}(x, y) = (-3x, -y)$. Calculate $\nabla \cdot \mathbf{F}$ and explain why your answer is consistent with your sketch.

Solution. Since $\nabla \cdot \mathbf{F} = (-3) + (-1) = -4$, we expect that the flow lines will take any particle towards the origin. In the slope field below, we have drawn in flow lines from every quadrant. It is apparent from the picture that the flow gravitates into the origin, which agrees with our calculation of the divergence:



Section 4.4, Problem 18: Calculate the scalar curl of the vector field $\mathbf{F}(x, y) = (y, -x)$.

Solution. The scalar curl is $(-1) - 1 = -2$.

Section 4.4, Problem 20: Calculate the scalar curl of the vector field $\mathbf{F}(x, y) = (x, y)$.

Solution. The scalar curl is $0 - 0 = 0$.

Section 4.4, Problem 26: Show that $\mathbf{F}(x, y) = (x^2 + y^2, -2xy)$ is not a gradient field.

Solution. If **F** is a gradient field, then its curl will be zero. The scalar curl of **F** is $-2y - 2y = -4y$. Since this is not identically zero, **F** can not be the gradient of any planar vector field.