20R Homework 4
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5.1.1.a: Evaluate $\int_{-1}^{1}\int_{0}^{1}(x^4y+y^2)\,dy\,dx$.

Solution:

$$\int_{-1}^{1}\int_{0}^{1}(x^4y+y^2)\,dy\,dx$$

$$=\int_{-1}^{1}\left[x^4\frac{y^2}{2}+\frac{y^3}{3}\right]_0^1 dx$$

$$=\int_{-1}^{1}\left(\frac{x^4}{2}+\frac{1}{3}\right)dx$$

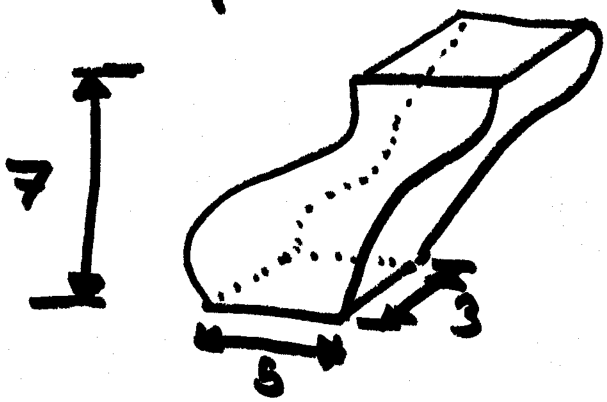
$$=\left[\frac{x^5}{10}+\frac{x}{3}\right]_{-1}^{1}$$

$$=\left(\frac{1}{10}+\frac{1}{3}\right)-\left(\frac{-1}{10}-\frac{1}{3}\right)$$

$$=\frac{2}{10}+\frac{2}{3}$$

$$=\frac{13}{15}$$

5.1.4: Compute the volume of



using Cavalieri's principle.

Solution: Each cross-section is a rectangle of area $5 \times 3 = 15$. Thus $A(x) = 15$. Thus the volume is

$$\int_0^7 A(x)\,dx = 15 \cdot 7 = 105.$$

5.2.2.d: Evaluate

$$\int\int_{[0,1]\times[0,1]} (x^2 + 2xy + y\sqrt{x})\,dx\,dy$$

Solution: Using Fubini's theorem we see that the above is equal to

$$= \int_0^1 \int_0^1 (x^2 + 2xy + y\sqrt{x})\,dx\,dy$$

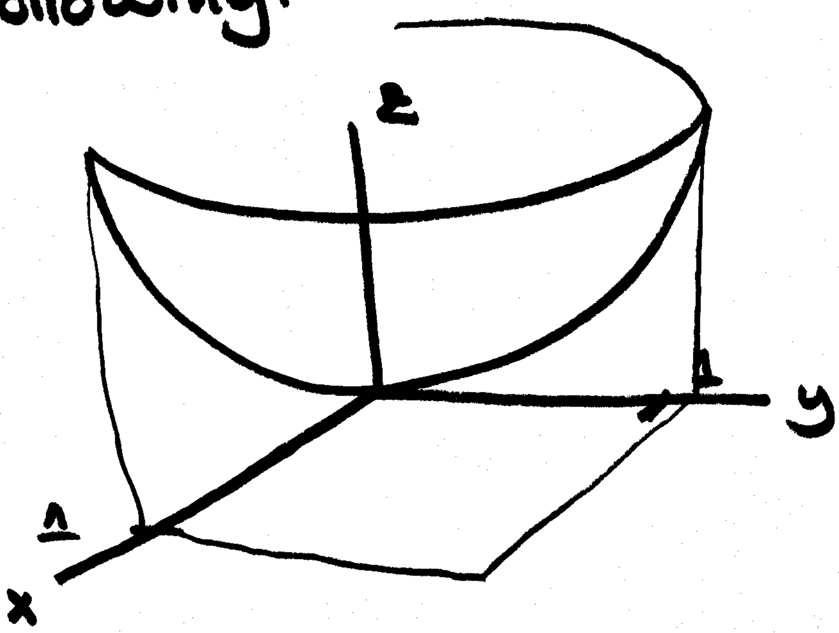
$$= \int_0^1 \left[\frac{x^3}{3} + x^2 y + y \frac{x^{3/2}}{3/2} \right]_{x=0}^{1} dy$$

$$= \int_0^1 \left(\frac{1}{3} + y + \frac{2y}{3} \right) dy$$

$$= \int_0^1 \left(\frac{1}{3} + \frac{5y}{3} \right) dy$$

$$= \left[\frac{y}{3} + \frac{5y^2}{6} \right]_0^1 = \frac{1}{3} + \frac{1}{6} = \frac{1}{2}.$$

S. 2.4: Compute the volume of the following:



thus the volume is

$$\iint_{[0,1]\times[0,1]} (x^2+y^2)\,dx\,dy$$

$$= \int_0^1 \int_0^1 (x^2+y^2)\,dx\,dy$$

$$= \int_0^1 \left[\frac{x^3}{3} + y^2 x \right]_{x=0}^{x=1} dy$$

$$= \int_0^1 \left(\frac{1}{3} + y^2 \right) dy$$

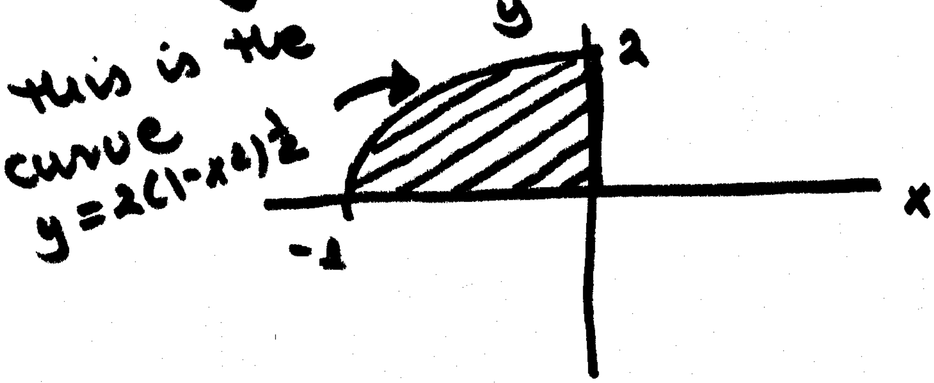
$$= \left[\frac{y}{3} + \frac{y^3}{3} \right]_0^1$$

$$= \frac{1}{3} + \frac{1}{3} = \frac{2}{3}.$$

5.3.2.f: Sketch the region of integration of the following integral and then evaluate it

$$\int_{-1}^{0} \int_0^{2(1-x^2)^{\frac{1}{2}}} x\,dy\,dx$$

Solution: In the middle integral y varies from 0 to $2(1-x^2)^{\frac{1}{2}}$, and in the outer one x varies from -1 to 0. Therefore the region of integration looks like



thus

$$\int_{-1}^{0}\int_{0}^{2(1-x^2)^{\frac{1}{2}}} x\,dy\,dx$$

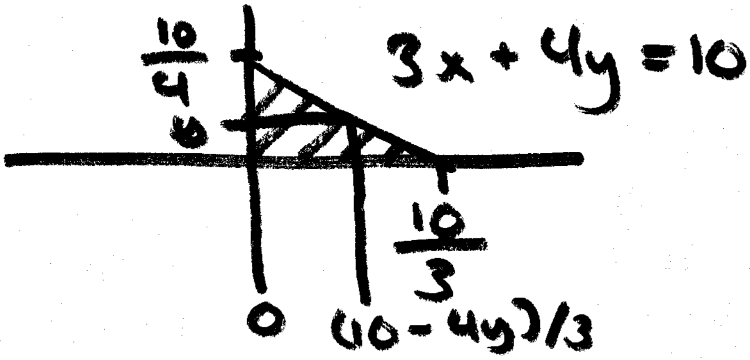
$$= \int_{-1}^{0} \Big[xy \Big]_{y=0}^{y=2(1-x^2)^{\frac{1}{2}}} dx$$

$$= \int_{-1}^{0} 2x(1-x^2)^{\frac{1}{2}}\,dx$$

$$= \left[2\frac{(-1)}{3}(1-x^2)^{3/2}\right]_{-1}^{0}$$

$$= \left(\frac{-2}{3}\right)\left[1-0\right] = -\frac{2}{3}$$

5.3.6: Let D be the region sketched below.



Evaluate $\iint_D (x^2+y^2)\,dx\,dy$.

Solution:

$$\iint_D (x^2+y^2)\,dx\,dy$$

$$= \int_0^{\frac{10}{4}} \int_0^{(10-4y)/3} (x^2+y^2)\,dx\,dy$$

$$= \int_0^{\frac{10}{4}} \left[\frac{x^3}{3} + xy^2 \right]_{x=0}^{x=(10-4y)/3} dy$$

$$= \int_0^{\frac{10}{4}} \left(\frac{1}{34} (10-4y)^3 + \frac{(10-4y)}{3} \right) dy$$

$$= \left[\frac{1}{3^4} \frac{(10-4y)^4}{(-16)} + \frac{(10-4y)^2}{(-24)} \right]_0^{\frac{10}{4}}$$

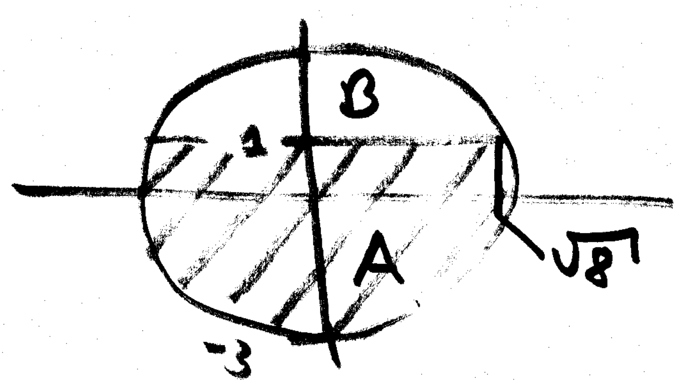
$$= \frac{-1}{3^4} \frac{10^4}{(-16)} - \frac{(10)^2}{(-24)}$$

= something...

5.4.2.b: Find

$$I = \int_{-3}^{1} \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} x^2 \, dx\, dy$$

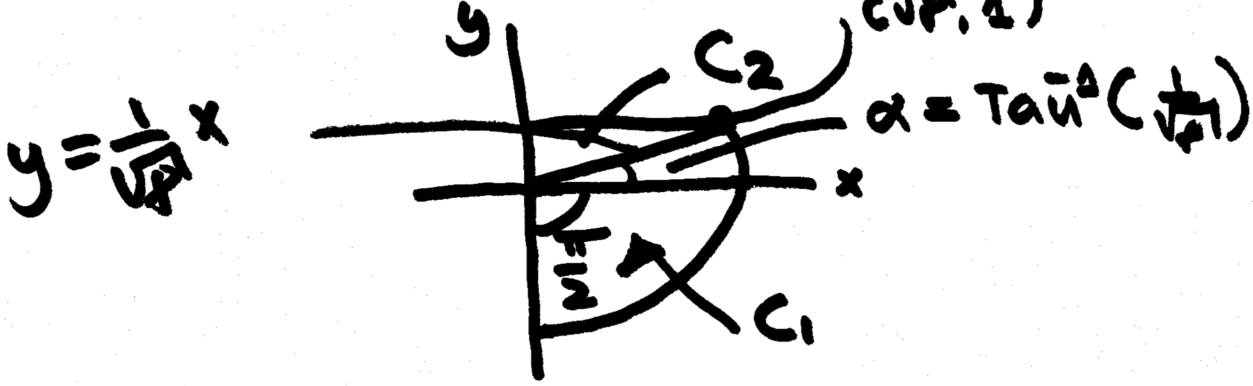
Solution: the area of integration looks as follows



First note that

$$I = 2\iint_C x^2\,dx\,dy,$$

where C is the region



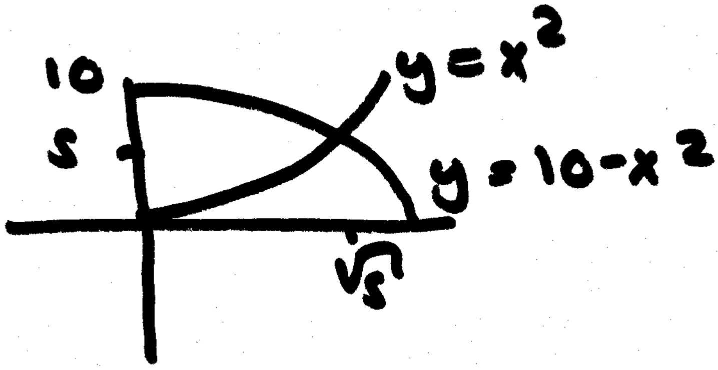
Also note that $\iint_C x^2\,dx\,dy = \iint_{C_1} x^2\,dx\,dy$ $+ \iint_{C_2} x^2\,dx\,dy$. In polars,

$$\iint_{C_1} x^2\,dx\,dy = \int_{r=0}^{3}\int_{-\frac{\pi}{2}}^{\alpha} r^2\cos^2\theta\; r\,dr\,d\theta$$

$$= \left[\int_0^3 r^3\,dr\right]\left[\int_{-\frac{\pi}{2}}^{\alpha} \frac{1}{2}\left((\cos 2\theta + 1\right)d\theta\right]$$

5.4.8: Compute $\int_D y^2\sqrt{x}\,dA$, where $D = \{(x,y) : x > 0,\ y > x^2,\ y < 10 - x^2\}$.

Solution: The region of integration looks like this



Thus

$$\int_D y^2\sqrt{x}\,dA = \int_0^{\sqrt{5}}\int_{x^2}^{10-x^2} y^2\sqrt{x}\,dy\,dx$$

$$= \int_0^{\sqrt{5}} \frac{\sqrt{x}}{3}\left[(10-x^2)^3 - x^6\right]dx$$

$$= \int_0^{\sqrt{5}} \frac{\sqrt{x}}{3}\left[1000 - 300x^2 + 30x^4 - 2x^6\right]dx$$

$$= \int_0^{\sqrt{8}} \frac{1}{3} \left(1000\sqrt{x} - 300x^{5/2} + 30x^{9/2} - 2x^{13/2} \right) dx \,.$$

$\cancel{= 2 \cdot S^{7/2} + \frac{4}{5} \cdot S \cdot \frac{2}{7} \cdot S^{7/2} + \frac{4}{5} \cdot \frac{2}{7} \cdot \frac{2}{9} \cdot S^{9/2}}$

~~= something...~~

S.S. 4: Evaluate

$$\iiint_{[0,1]^3} z e^{x+y} \, dx\,dy\,dz$$

$$= \int_{[0,1]} \int_{[0,1]} \int_{[0,1]} z \cdot e^x \cdot e^y \, dx\,dy\,dz$$

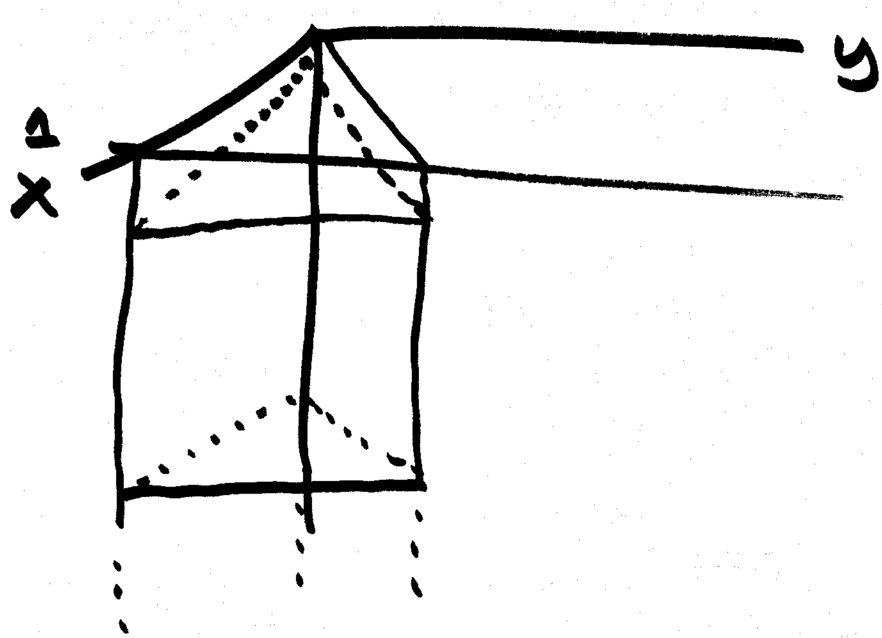
$$= \left[\int_{[0,1]} z \, dz \right] \left[\int_{[0,1]} e^y \right] \left[\int_{[0,1]} e^x \, dx \right]$$

$$= \left[\frac{z^2}{2} \right]_0^1 \left[e^y \right]_0^1 \left[e^x \right]_0^1$$

$$= \frac{1}{2} (e^1 - 1)(e^1 - 1)$$

S.S. 11: Find the volume of the solid bounded by $y=x$, $z=0$ $y=0$, $x=1$ and $x+y+z=0$.

Solution: the solid is impossible to draw but before the plane is added the solid looks like a Toblerone bar



The prism then gets cut by

the plane at an angle. The volume is therefore

$$\int\int\int_{V} dx\,dy\,dz$$

$$= \int_0^1 \int_0^x \int_0^{-x-y} dz\,dy\,dx$$

$$= \int_0^1 \int_0^x (-x-y)\,dy\,dx$$

$$= \int_0^1 \left[-xy - \frac{y^2}{2}\right]_0^x dx$$

$$= \int_0^1 -\frac{3x^2}{2}\,dx$$

$= -\frac{1}{2}$. So I guess the volume is $\frac{1}{2}$.