Section 6.1, Problem 2: Define $T(x,y) = \left(\frac{x-y}{\sqrt{2}}, \frac{x+y}{\sqrt{2}}\right)$. Show that T rotates the unit square $[0,1] \times [0,1]$.

Solution. First of all, since

$$\begin{aligned}\sqrt{\left(\frac{x-y}{\sqrt{2}}\right)^2 + \left(\frac{x+y}{\sqrt{2}}\right)^2} &= \sqrt{\frac{(x^2+2xy+y^2)+(x^2-2xy+y^2)}{2}} \\ &= \sqrt{\frac{2x^2+2y^2}{2}} \\ &= \sqrt{x^2+y^2},\end{aligned}$$

T preserves magnitude. Let θ be the angle between (x,y) and $T(x,y)$. Then

$$\begin{aligned}\cos\theta &= \frac{(x,y)\cdot\left(\frac{x-y}{\sqrt{2}}, \frac{x+y}{\sqrt{2}}\right)}{\|(x,y)\| \cdot \|\left(\frac{x-y}{\sqrt{2}}, \frac{x+y}{\sqrt{2}}\right)\|} \\ &= \frac{\frac{x^2-xy}{\sqrt{2}} + \frac{xy+y^2}{\sqrt{2}}}{x^2+y^2} \\ &= \frac{1}{\sqrt{2}} \cdot \frac{x^2+y^2}{x^2+y^2} \\ &= \frac{1}{\sqrt{2}}.\end{aligned}$$

Since T sends every point (x,y) to one of equal magnitude at an angle of θ counterclockwise, T is a rotation of the entire square.

Section 6.1, Problem 3: Let $D^* = [0,1] \times [0,1]$ and define T on D^* by $T(u,v) = (-u^2 + 4u, v)$. Find the image D. Is T one-to-one?

Solution. Since both $-u^2+4u$ and v are increasing functions on $[0,1]\times[0,1]$, the endpoints of the range are determined by the endpoints of D^*. So $D = [0,3] \times [0,1]$.

Next, we show that T is one-to-one. To show this, suppose $T(x,y) = T(u,v)$. Then $(-x^2 + 4x, y) = (-u^2+4u, v)$, which implies that $y = v$ and $-x^2+4x = -u^2+4u$. Rearranging terms, we have

$$\begin{aligned}-x^2+4x &= -u^2+4u \\ u^2 - x^2 + 4x - 4u &= 0 \\ (u-x)(u+x) + 4(x-u) &= 0 \\ (u-x)(u+x-4) &= 0,\end{aligned}$$

so either $u = x$ or $u + x = 4$. Since u and x are in D^*, they are both between 0 and 1. So we cannot have $u + x = 4$, which implies that $u = x$. So we can conclude that T is one-to-one.

Section 6.1, Problem 4: Let D^* be the parallelogram bounded by the lines $y = 3x - 4$, $y = 3x$, $y = \frac{1}{2}x$, and $y = \frac{1}{2}(x+4)$. Let $D = [0,1]\times[0,1]$. Find a T such that D is the image of D^* under T.

Solution. We need the corners of D^* to be sent to the corners of D. Set $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The corners of D^* are $(0,0)$, $\left(\frac{8}{5}, \frac{4}{5}\right)$, $\left(\frac{12}{5}, \frac{16}{5}\right)$, and $\left(\frac{4}{5}, \frac{12}{5}\right)$.

Setting $T(8/5,4/5) = (1,0)$, $T(4/5,12/5) = (0,1)$ gives

$$\begin{aligned} 8a + 4b &= 5 \\ 8c + 4d &= 0 \\ 4a + 12b &= 0 \\ 4c + 12d &= 5 \end{aligned}$$

Solving yields $a = 3/4$, $b = -1/4$, $c = -1/4$, and $d = 1/2$, so $T = \begin{pmatrix} 3/4 & -1/4 \\ -1/4 & 1/2 \end{pmatrix}$ is a valid transformation.

Section 6.1, Problem 10: Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is linear and is given by $T(\mathbf{x}) = A\mathbf{x}$, where A is a 2×2 matrix. Show that if det $A \neq 0$, then T takes parallelograms onto parallelograms.

Solution. Say we have a parallelogram P. It can be described as the set of points of the form $\mathbf{p} + \lambda\mathbf{v} + \mu\mathbf{w}$, where $\mathbf{v}$ and $\mathbf{w}$ are not scalar multiples of each other and $\lambda, \mu \in [0,1]$. Say $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $T(P)$ is the set of points of the form $A\mathbf{p} + \lambda A\mathbf{v} + \mu A\mathbf{w}$. Since det $A \neq 0$, $A\mathbf{v}$ and $A\mathbf{w}$ are not scalar multiples of each other. So T transforms parallelograms into parallelograms.

Section 6.2, Problem 1: Let D be the unit disk $x^2 + y^2 \leq 1$. Evaluate

$$\int\int_D e^{x^2+y^2}\, dxdy.$$

Solution. Transferring to polar coordinates, our range is $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$:

$$\begin{aligned} \iint_D e^{x^2+y^2}\, dxdy &= \int_0^r \int_0^{2\pi} re^{r^2}\, d\theta dr \\ &= 2\pi \int_0^r re^{r^2}\, dr \\ &= 2\pi \left(\left. \frac{e^{r^2}}{2} \right|_0^r \right) \\ &= \pi(e^{r^2} - 1). \end{aligned}$$

Section 6.2, Problem 4(a): Let D^* be the rectangle $[0,1] \times [1,2]$, $T(u,v) = (u, v + uv)$, and $D = T(D^*)$. Find D and evaluate $\iint_D xy\, dxdy$.

Solution. D is the region bounded by $x = 0$, $x = 1$, $y = x + 1$, and $y = 2x + 2$. The Jacobian matrix is $\begin{pmatrix} 1 & 0 \\ v & 1+u \end{pmatrix}$, and its determinant is $1 + u$, so our change-of-basis integral is

$$\begin{aligned} \iint_D xy\, dxdy &= \int_0^1 \int_1^2 (uv + u^2v) \cdot (1+u)\, dvdu \\ &= \int_0^1 \int_1^2 v(u^3 + 2u^2 + u)\, dvdu \\ &= \frac{3}{2} \int_0^1 u^3 + 2u^2 + u\, du \\ &= \frac{3}{2} \left(\frac{1}{4} + \frac{2}{3} + \frac{1}{2} \right) \\ &= \frac{17}{8}. \end{aligned}$$

Section 6.2, Problem 5: Evaluate $\iint_D \frac{dxdy}{\sqrt{1+x+2y}}$, where $D = [0,1] \times [0,1]$, by setting $T(u,v) = (u, v/2)$ and evaluating an integral over D^*. where $T(D^*) = D$.

Solution. Using $T^{-1}(u,v) = (u, 2v)$, we get the bounds for the integral are $[0,1] \times [0,2]$. The determinant of the Jacobian is $1/2$, so the integral evaluates to

$$\begin{aligned}\iint_D \frac{dxdy}{\sqrt{1+x+2y}} &= \int_0^1 \int_0^2 \frac{dvdu}{2\sqrt{1+u+v}} \\ &= \int_0^1 \sqrt{3+u} - \sqrt{1+u}\, du \\ &= \frac{2}{3}(8 - 3\sqrt{3}) - \frac{2}{3}(2\sqrt{2} - 1) \\ &= \frac{2}{3}(9 - 3\sqrt{3} - 2\sqrt{2}).\end{aligned}$$

Section 6.2, Problem 8: Calculate $\iint_R \frac{1}{x+y} dxdy$ where R is the region bounded by $x = 0$, $y = 0$, $x + y = 1$, and $x + y = 4$, by using the mapping $T(u,v) = (u - uv, uv)$.

Solution. The bounds on u and v for T are $1 \leq u \leq 4$ and $0 \leq v \leq 1$. This can be seen by first noticing that the boundaries $x + y = 1$ and $x + y = 4$ are satisfied by $u = (u - uv) + uv = u$ and $u = (u - uv) + uv = 4$. After we have the bounds on u, substitute in to get the bounds on v.

The Jacobian matrix is $\begin{pmatrix} 1-v & -u \\ v & u \end{pmatrix}$, and its determinant is $(1-v)u + uv = u$, so the integral equals

$$\begin{aligned}\iint_R \frac{1}{x+y} dxdy &= \int_1^4 \int_0^1 \frac{1}{u} \cdot u\, dvdu \\ &= \int_1^4 du \\ &= 3.\end{aligned}$$

Section 6.2, Problem 9: Evaluate $\iint_D (x^2 + y^2)^{3/2}\, dxdy$ where D is the disk $x^2 + y^2 \leq 4$.

Solution. We will change to polar coordinates, where our ranges are $0 \leq r \leq 2$ and $0 \leq \theta \leq 2\pi$. The change-of-variables determinant is r, so the integral evaluates to

$$\begin{aligned}\iint_D (x^2 + y^2)^{3/2}\, dxdy &= \int_0^2 \int_0^{2\pi} (r^2)^{3/2} \cdot r\, d\theta dr \\ &= \int_0^2 \int_0^{2\pi} r^4\, d\theta dr \\ &= 2\pi \int_0^2 r^4\, dr \\ &= \frac{64\pi}{5}.\end{aligned}$$

Section 6.2, Problem 19: Integrate $x^2 + y^2 + z^2$ over the cylinder $x^2 + y^2 \leq 2$, $-2 \leq z \leq 3$.

Solution. The ranges for the cylindrical coordinates of this region are $0 \leq r \leq \sqrt{2}$,

$0 \leq \theta \leq 2\pi$, and $-2 \leq z \leq 3$. The determinant of the Jacobian matrix is r, so our integral evaluates to

$$\begin{aligned}\iiint_V x^2+y^2+z^2\,dxdydz &= \int_{-2}^{3}\int_0^{2\pi}\int_0^{\sqrt{2}}(r^2+z^2)\cdot r\,drd\theta dz\\ &= \int_{-2}^{3}\int_0^{2\pi} 1+z^2\,d\theta dz\\ &= 2\pi\int_{-2}^{3}1+z^2\,dz\\ &= \frac{100\pi}{3}.\end{aligned}$$

Section 6.2, Problem 29: Let E be the ellipsoid $(x/a)^2+(y/b)^2+(z/c)^2 \leq 1$, where a, b, and c are positive.

(a) Find the volume of E.
(b) Evaluate $\iiint_E[(x/a)^2+(y/b)^2+(z/c)^2]\,dxdydz$.

Solution.

(a) The change of variables we enact will be $T(u,v,w) = (au, bv, cw)$, where out range is $0 \leq u^2+v^2+w^2 \leq 1$, the unit sphere, which we will denote as S^2. The determinant of the Jacobian is abc. The volume is

$$\begin{aligned}\iiint_E dxdydz &= \iiint_{S^2} abc\,dudvdw\\ &= \frac{4\pi}{3}\cdot abc = \frac{4\pi abc}{3}.\end{aligned}$$

(b) We start off with our change of variables from last time:

$$\iiint_E (x/a)^2+(y/b)^2+(z/c)^2\,dxdydz = abc\iiint_{S^2} u^2+v^2+w^2\,dudvdw$$

Now we switch to spherical coordinates – our ranges are $0 \leq \rho \leq 1$, $0 \leq \theta \leq 2\pi$, and $0 \leq \phi \leq \pi$. The determinant of the change of variables is $\rho^2 \sin\phi$, so our integral is

$$\begin{aligned}abc\iiint_{S^2} u^2+v^2+w^2\,dudvdw &= abc\int_0^{2\pi}\int_0^{\pi}\int_0^1 \rho^2\cdot\rho^2\sin\phi\,d\rho d\phi d\theta\\ &= \frac{abc}{5}\int_0^{2\pi}\int_0^{\pi}\sin\phi\,d\phi d\theta\\ &= \frac{2abc}{5}\int_0^{2\pi}d\theta\\ &= \frac{4\pi abc}{5}.\end{aligned}$$

Section 7.1, Problem 2(a): Evaluate the path integral $\int_{\mathbf{c}} f(x,y,z)\,ds$, where $f(x,y,z) = x+y+z$ and $\mathbf{c}(t) = (\sin t, \cos t, t)$, $t \in [0, 2\pi]$.

Solution.

$$\begin{aligned}\int_{\mathbf{c}} f(x,y,z)\,ds &= \int_0^{2\pi}(\sin t+\cos t+t)\cdot\sqrt{\cos^2 t+(-\sin t)^2+1^2}\,dt\\ &= \sqrt{2}\int_0^{2\pi}\sin t+\cos t+t\,dt\\ &= 2\sqrt{2}\pi^2.\end{aligned}$$

Section 7.1, Problem 4(a): Evaluate the path integral of $f(x,y,z)$, where $f(x,y,z) = x\cos z$, $\mathbf{c}(t) = (t, t^2, 0)$, $t \in [0,1]$.

Solution.

$$\begin{aligned}\int_{\mathbf{c}} f(x,y,z)\ ds &= \int_0^1 t\cos 0 \cdot \sqrt{1+4t^2}\ dt \\ &= \int_0^1 t\sqrt{1+4t^2}\ dt \\ &= \left.\frac{(1+4t^2)^{3/2}}{12}\right|_0^1 \\ &= \frac{5\sqrt{5}-1}{12}.\end{aligned}$$

Section 7.2, Problem 1(c): Evaluate the line integral of $\mathbf{F}(x,y,z) = (x,y,z)$ over the path $\mathbf{c}(t) = (\sin t, 0, \cos t)$, $0 \le t \le 2\pi$.

Solution.

$$\begin{aligned}\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} &= \int_0^{2\pi} (\sin t, 0, \cos t) \cdot (\cos t, 0, -\sin t)\ dt \\ &= \int_0^{2\pi} 0\ dt \\ &= 0.\end{aligned}$$

Section 7.2, Problem 2(a): Evaluate the line integral $\int_{\mathbf{c}} x\ \ dy - y\ \ dx$, where $\mathbf{c}(t) = (\cos t, \sin t)$, $0 \le t \le 2\pi$.

Solution.

$$\begin{aligned}\int_{\mathbf{c}} x\ dy - y\ dx &= \int_0^{2\pi} \cos t \cdot (\cos t) - \sin t \cdot (-\sin t)\ dt \\ &= \int_0^{2\pi} 1\ dt \\ &= 2\pi.\end{aligned}$$

Section 7.2, Problem 3: Consider the force field $\mathbf{F}(x,y,z) = (x,y,z)$. Compute the work done in moving a particle along the parabola $y = x^2$, $z = 0$ from $x = -1$ to $x = 2$.

Solution. The path can be parameterized by $(t, t^2, 0)$ as $-1 \le t \le 2$, so the work done is

$$\begin{aligned}\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} &= \int_{-1}^{2} (t, t^2, 0) \cdot (1, 2t, 0)\ dt \\ &= \int_{-1}^{2} t + 2t^3\ dt \\ &= 9.\end{aligned}$$

Section 7.2, Problem 12: Let $\mathbf{F} = (z^3 + 2xy, x^2, 3xz^2)$. Show that the integral of $\mathbf{F}$ around the circumference of the square with vertices $(\pm 1, \pm 1, 0)$ is zero.

Solution. Call the circumference C. Looking from above, let the right, top, left, and bottom sides be R, T, L, and B. Then

$$\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{s} &= \int_L \mathbf{F}\cdot d\mathbf{s} + \int_T \mathbf{F}\cdot d\mathbf{s} + \int_R \mathbf{F}\cdot d\mathbf{s} + \int_B \mathbf{F}\cdot d\mathbf{s} \\
&= \int_{-1}^{1} (2t,1,0)\cdot(0,1,0)\,dt + \int_{1}^{-1} (2t,t^2,0)\cdot(1,0,0)\,dt \\
&\quad + \int_{1}^{-1} (-2t,-1,0)\cdot(0,1,0)\,dt + \int_{-1}^{1} (-2t,t^2,0)\cdot(1,0,0)\,dt \\
&= \int_{-1}^{1} 2t - 2t + 2t - 2t\,dt \\
&= \int_{-1}^{1} 0\,dt \\
&= 0.
\end{aligned}$$

Section 7.2, Problem 15: Evaluate the integral

$$\int_C 2xyz\,dx + x^2\,dy + x^2y\,dz,$$

where C is an oriented simple curve connecting $(1,1,1)$ to $(1,2,4)$.

Solution. Since this force field is the gradient of the function $f(x,y,z) = x^2yz$, we have $\int_{\mathbf{c}} \mathbf{F}\cdot d\mathbf{s} = f(\mathbf{c}(b)) - f(\mathbf{c}(a))$ for *any* simple oriented curve $\mathbf{c}$ from $\mathbf{c}(a)$ to $\mathbf{c}(b)$. So all we need is f and the endpoints! The integral is then

$$f(1,2,4) - f(1,1,1) = 7.$$

Section 7.2, Problem 18: Ugh, story problems... a cyclist rides up a mountain along the path shown in figure 7.2.16. She makes one complete revolution around the mountain in reaching the top, while her vertical rate of climb is constant. Throughout the trip she exerts a force described by the vector field $\mathbf{F}(x,y,z) = (y,x,1)$.

What is the work done by the cyclist in travelling from A to B? What is unrealistic about this model of a cyclist?

Solution. Since the vector field $\mathbf{F}$ is the gradient of the function $f(x,y,z) = xy + z^2$, the total work done can be calculated by calculating $f(\mathbf{c}(b)) - f(\mathbf{c}(a))$, where $\mathbf{c}(a)$ and $\mathbf{c}(b)$ are the endpoints of the path. By the picture, we get the endpoints to be $(0,\sqrt{2\pi},0)$ and $(0,0,2\pi)$. So the total work done is

$$\int_{\mathbf{c}} \mathbf{F}\cdot d\mathbf{s} = f(0,0,2\pi) - f(0,\sqrt{2\pi},0) = 2\pi.$$

As for realism, we should note that if the cyclist takes a path that winds around and around the mountain over and over on its way up will encounter a lot more friction (on the ground, and also internally in the bike), and hence a longer path may very well require more work than a shorter one.