

MATH 20E VECTOR CALCULUS

Lecture 1: Overview+Review vector operations.

This course is given after a first course in multivariable calculus.

What is the previous multivariable calculus course about?

Curves in space $(x(t), y(t), z(t))$, e.g. path of a particle.

Vectors and vector operations, e.g. the dot and the cross product.

Equations of lines and planes. (using vectors)

The the velocity $\mathbf{v}(t) = (x'(t), y'(t), z'(t))$, tangent vector to the particle.

Functions of several variables $f(x, y, z)$, e.g. temperature at each point in space.

Derivatives of functions of several variables:

E.g. if f represent the temperature then the gradient vector $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$

represents the direction in which the temperature increases the most.

Multiple integrals: Double integrals over domains in the plane and triple or volume integrals over domains in space. Polar coordinates.

What is this vector calculus course about?

Ch 1: Vectors and vector operations. (review of 20C)

Ch 2: Derivatives of functions of several variables. (review of 20C)

Ch 3: Taylor's theorem for functions of several variables. (skip min/max)

Ch 4: **Vector fields** $\mathbf{v}(x, y, z) = ((v_1(x, y, z), v_2(x, y, z), v_3(x, y, z)))$, at each point in space we are given a vector, and the **divergence** and **curl** of them.

The gravitational field is a vector field; at each point in space we are given a vector.

The vector field of a continuum of fluid particles is a vector field. At each point in space we are given the velocity of the particles at that point.

Each component of a vector field can be differentiated in each direction.

The divergence and curl are special derivatives with physical meaning.

Ch 5 Double and Triple Integrals (review of 20C)

Ch 6 The **change of variable theorem** in multiple integrals.

Ch 7 **Integrals over paths and surfaces, surface area, flow through surface.**

Ch 8: **The Divergence Theorem, Greens Theorem and Stokes Theorem.**

Generalization of the fundamental theorem of calculus $\int_a^b f'(x) dx = f(b) - f(a)$

to several variables: E.g. $F = (F_1, F_2, 0)$ a vector function and D a domain in the

$x - y$ plane with boundary C then $\int_C F_1 dx + F_2 dy = \iint_D (\partial F_2 / \partial x - \partial F_1 / \partial y) dx.$

Surface integrals is the hardest part. Thinking geometrically and physically helps understanding concepts but is not needed to do problems

Section 1.1 Vectors.

Recall that a **point** in space can be represented by an ordered triplet (x, y, z) of real numbers called the **Cartesian or rectangular coordinates**, that measure the lengths of the projections on the three coordinate axis.

A **vector** \mathbf{a} is a directed line segment or arrow; it has a length $\|\mathbf{a}\|$ and a direction. Its components (a_1, a_2, a_3) are the coordinates of the endpoint if it starts at the origin. In the notes vectors will be denoted by boldface letters \mathbf{a} and in the lectures by \vec{a} .

Addition of vectors is geometrically defined by the triangle law and algebraically by $(a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$.

Scalar multiplication $\alpha \mathbf{a}$ is a vector in the same direction as \mathbf{a} with length $|\alpha| \|\mathbf{a}\|$ and its algebraically given by $\alpha(a_1, a_2, a_3) = (\alpha a_1, \alpha a_2, \alpha a_3)$.

The **length** of the vector is $\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$, by the Pythagorean theorem.

A **unit vector** (i.e. of length one) in the direction of the vector \mathbf{a} is given by $\frac{\mathbf{a}}{\|\mathbf{a}\|}$.

Standard basis: If $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$ and $\mathbf{k} = (0, 0, 1)$ then

$$\mathbf{a} = (a_1, a_2, a_3) = a_1(1, 0, 0) + a_2(0, 1, 0) + a_3(0, 0, 1) = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}.$$

If $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ are points in space then the vector represented by the directed line segment $\overrightarrow{P_1P_2}$ is $(x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}$.

What can we do with vectors?

Ex Find the parametric equations of a line passing through $(0, 1, 2)$ and $(1, 2, 4)$.

Sol The vector $\mathbf{v} = (1, 2, 4) - (0, 1, 2) = (1, 1, 2)$ is parallel to the line and so is any multiple $t\mathbf{v}$. The points on the line are therefore given by $(x, y, z) = (0, 1, 2) + t(1, 1, 2) = (t, 1 + t, 2 + 2t)$.

We didn't have time to cover the rest so its homework to read it:.

Section 1.2: Inner product (or scalar or dot product) of two vectors is

$$(1) \quad \mathbf{a} \cdot \mathbf{b} = (a_1, a_2, a_3) \cdot (b_1, b_2, b_3) = a_1b_1 + a_2b_2 + a_3b_3$$

The geometric interpretation is $\|\mathbf{a}\| \|\mathbf{b}\| \cos \theta$, where θ is the angle between \mathbf{a} and \mathbf{b} . Note that $\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2$ and that $\mathbf{a} \cdot \mathbf{b} = 0$ if and only if \mathbf{a} and \mathbf{b} are perpendicular. Note that $\|\mathbf{b}\| \cos \theta$ is the component of \mathbf{b} in the direction of \mathbf{a} .

The orthogonal projection \mathbf{p} of \mathbf{b} on \mathbf{a} is given by $\mathbf{p} = \|\mathbf{b}\| \cos \theta \frac{\mathbf{a}}{\|\mathbf{a}\|} = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|^2} \mathbf{a}$.

Ex. Decompose $\mathbf{b} = 2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ into a vector \mathbf{b}_{\parallel} parallel to $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and a vector \mathbf{b}_{\perp} perpendicular to \mathbf{a} .

Sol. $\mathbf{a} \cdot \mathbf{b} = 2 - 2 + 4 = 4$, $\|\mathbf{a}\| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$ so $\mathbf{b}_{\parallel} = (\mathbf{a} \cdot \mathbf{b}) \frac{\mathbf{a}}{\|\mathbf{a}\|^2} = 4(\mathbf{i} + \mathbf{j} + \mathbf{k})/3$ and $\mathbf{b}_{\perp} = \mathbf{b} - \mathbf{b}_{\parallel} = (2 - 4/3)\mathbf{i} - (1 + 4/3)\mathbf{j} + (4 - 4/3)\mathbf{k}$.