

Lecture 11: Changing the order of integration.

Ex. Evaluate $\int_0^1 \int_{3y}^3 e^{x^2} dx dy$.

$$\int_0^1 \int_{3y}^3 e^{x^2} dx dy = \iint_D e^{x^2} dA, \quad D = \{(x, y); 0 \leq y \leq 1, 3y \leq x \leq 3\}$$

Here

$$D = \{(x, y); 0 \leq y \leq 1, 3y \leq x \leq 3\} = \{(x, y); 0 \leq x \leq 3, 0 \leq y \leq x/3\}$$

so

$$\iint_D e^{x^2} dA = \int_0^3 \int_0^{x/3} e^{x^2} dy dx = \int_0^3 e^{x^2} y \Big|_{y=0}^{x/3} dx = \int_0^3 e^{x^2} \frac{x}{3} dx = \frac{e^{x^2}}{6} \Big|_{x=0}^3 = \frac{e^9 - 1}{6}$$

Areas can be calculated using double integrals:

$$\iint_D 1 dA = \text{Area}(D).$$

This is because the integral is the volume above D and below 1 which is $\text{Area}(D) \cdot 1$.

Triple Integrals. We now want to define the integral of a function f over a rectangular box $B = \{(x, y, z); a \leq x \leq b, c \leq y \leq d, r \leq z \leq s\}$. We divide B into smaller boxes $B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$ by dividing the interval $[a, b]$ into n subintervals of length $\Delta x = (b - a)/n$, $[c, d]$ into n subintervals of length Δy and $[r, s]$ into n subintervals of length Δz . Then we form the Riemann sum

$$(1) \quad \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*) \Delta V$$

where $(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)$ is a sample point in B_{ijk} and $\Delta V = \Delta x \Delta y \Delta z$. We then define the integral to be the limit of the Riemann sum:

$$(2) \quad \iiint_B f(x, y, z) dV = \lim_{n \rightarrow \infty} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*) \Delta V$$

Triple integrals do not have the same direct geometric interpretation as double integrals and volumes, because it is difficult to visualize four dimensional volumes. However, if $f(x, y, z)$ represent the density of mass per unit volume at a point (x, y, z) then the integral over B of f is the total mass of B .

As for the case of two variables we can write it as iterated integrals (Fubini):

$$(3) \quad \iiint_B f(x, y, z) dV = \int_r^s \left(\int_c^d \left(\int_a^b f(x, y, z) dx \right) dy \right) dz$$

or one can integrate the different variables in any other order.

Ex. Let $B = \{(x, y, z); 1 \leq x \leq 2, 0 \leq y \leq 1, 1 \leq z \leq 2\}$. Find $\iiint_B x^2 y z dV$.

Sol. $\iiint_B x^2 y z dV = \int_1^2 \int_1^2 \int_0^1 x^2 y z dy dz dx = \dots$

As for double integrals we define the integral of f over a more general bounded region E by finding a large box B containing E and integrating the function that is equal to f in E and 0 outside E over the larger box B .

We now restrict our attention to some special regions. Region of **type 1**:

$$(4) \quad E = \{(x, y, z); (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\}$$

where D is the projection of E onto the x - y plane. Then

$$(5) \quad \iiint_E f(x, y, z) dV = \iint_D \left(\int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z) dz \right) dA$$

In particular if D is a region of type I in the plane then

$$(6) \quad E = \{(x, y, z); a \leq x \leq b, g_1(x) \leq y \leq g_2(x), u_1(x, y) \leq z \leq u_2(x, y)\}$$

and

$$(7) \quad \iiint_E f(x, y, z) dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z) dz dy dx$$

Note also that one can find the volume by using triple integrals

$$(8) \quad \text{Volume}(E) = \iiint_E 1 dV$$

Ex. Let $E = \{(x, y, z); x \geq 0, y \geq 0, z \geq 0, x + 2y + 3z \leq 1\}$. Find the volume of E

Sol. $E = \{(x, y, z); 0 \leq x \leq 1, 0 \leq y \leq (1-x)/2, 0 \leq z \leq (1-x-2y)/3\}$ so

$$V = \iiint_E 1 dV = \int_0^1 \int_0^{(1-x)/2} \int_0^{(1-x-2y)/3} dz dy dx = \dots$$

Region of type 2: If $E = \{(x, y, z); (y, z) \in D, u_1(y, z) \leq x \leq u_2(y, z)\}$. then

$$\iiint_E f(x, y, z) dV = \iint_D \left(\int_{u_1(y, z)}^{u_2(y, z)} f(x, y, z) dx \right) dA$$

Region of type 3: If $E = \{(x, y, z); (x, z) \in D, u_1(x, z) \leq y \leq u_2(x, z)\}$ then

$$\iiint_E f(x, y, z) dV = \iint_D \left(\int_{u_1(x, z)}^{u_2(x, z)} f(x, y, z) dy \right) dA$$

Ex. Evaluate $\iiint_E \sqrt{x^2 + z^2} dV$ where E is the region bounded by $y = x^2 + z^2$ and the plane $y = 4$.

Sol. $E = \{(x, y, z); (x, z) \in D, x^2 + z^2 \leq y \leq 4\}$, where $D = \{(x, z); x^2 + z^2 \leq 4\}$.

$$\iiint_E \sqrt{x^2 + z^2} dV = \iint_D \int_{x^2+z^2}^4 \sqrt{x^2 + z^2} dy dA = \iint_D (4 - (x^2 + z^2)) \sqrt{x^2 + z^2} dA$$

Introducing polar coordinates in the x - z plane: $x = r \cos \theta$, $z = r \sin \theta$, $dA = dx dz = r dr d\theta$ gives that the integral is: $\int_0^{2\pi} \int_0^2 (4 - r^2) r r dr d\theta = \dots = \frac{128\pi}{15}$