Lecture 12: The change of variable theorem on the line. Suppose that $x(u)$, $a \le u \le b$ is a change of variables. In order for it to be invertible we assume that $dx(u)/du > 0$, when $a \le u \le b$. Then we can change variables in the integral:

$$\int_{x(a)}^{x(b)} f(x)\,dx = \int_a^b f(x(u))\frac{dx}{du}du \tag{1}$$

i.e. symbolically

$$dx = \frac{dx}{du}du.$$

A small change $\triangle u$ gives a small change $\triangle x \sim x'(u)\triangle u$, by the linear approximation.

We will give similar theorem for functions of two variables. Let $T(u,v) = (x,y)$;

$$x = x(u,y), \qquad y = y(u,v), \tag{2}$$

be a **mapping** from a piecewise smooth simply connected domain D^* in the u-v plane **onto** a simply connected domain $D = T(D^*)$ in the x-y plane. We say that a map is **one-to-one** or **invertible** if no two points are mapped to one point, i.e. if $T(u,v) = T(u',v')$ implies that $(u,v) = (u',v')$. In order for the map to be invertible we assume that the **Jacobian determinant**

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \neq 0, \tag{3}$$

is non-vanishing everywhere, see the **Inverse Function Theorem** in section 3.5. The condition that the determinant of the derivative map DT is non-vanishing is equivalent to that the directional derivative is non-vanishing in any direction, since the directional derivative $\mathbf{T}_u\triangle u+\mathbf{T}_v\triangle v$ is non-vanishing in any direction $(\triangle u, \triangle v)$ if and only if the column vectors $\mathbf{T}_u = \begin{bmatrix} \partial x/\partial u \\ \partial y/\partial u \end{bmatrix}$ and $\mathbf{T}_v = \begin{bmatrix} \partial x/\partial v \\ \partial y/\partial v \end{bmatrix}$ are not parallel.

Ex Let $D^* = \{(r,\theta);\ 0<r<1,\ 0\le\theta<2\pi\}$ and let $(x,y) = T(r,\theta) = (r\cos\theta, r\sin\theta)$. Then $D = T(D^*) = \{(x,y);\ 0<x^2+y^2\le 1\}$, since $x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2 \le 1$, so $T(D^*)\subset D$ and any point in D can be written as $(r\cos\theta, r\sin\theta)$. Moreover, $T(r,\theta) = T(r',\theta')$ implies that $(r,\theta) = (r',\theta')$, if both points are in D^*.

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & r\sin\theta \\ \sin\theta & -r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r \neq 0.$$

Change of variable theorem in the plane.

$$\iint_D f(x,y)\,dxdy = \iint_{D^*} f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv \tag{4}$$

If $f=1$ then the left is the area of the domain D. We will argue that the Jacobian gives the local change of area scale under the mapping, symbolically

$$dxdy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv \tag{5}$$

The geometry of maps of the plane To make sense of (5) we study a linear map:

$$\text{(6)} \qquad \begin{cases} x = au + bv, \\ y = cu + dv, \end{cases} \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

for which the Jacobian determinant is a constant

$$\text{(7)} \qquad \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = \det A.$$

Let R^* be a small rectangle in the u-v plane with side lengths $\triangle u$ and $\triangle v$

$$\text{(8)} \qquad R^* = \{(u,v);\ 0 \le u \le \triangle u,\ 0 \le v \le \triangle v\}$$

We claim that the image of R^* in the x-y plane is a small parallelogram R with adjacent sides formed by the two vectors $(a\mathbf{i} + c\mathbf{j})\triangle u$ and $(b\mathbf{i} + d\mathbf{j})\triangle v$. To see this we write (6) as vector equation in matrix notation

$$\text{(9)} \qquad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} u + \begin{bmatrix} b \\ d \end{bmatrix} v, \qquad \text{where} \qquad \begin{bmatrix} x \\ y \end{bmatrix} = x\mathbf{i} + y\mathbf{j} \quad \text{etc.}$$

which is just another way of saying that $x\mathbf{i} + y\mathbf{j} = (a\mathbf{i} + c\mathbf{j})u + (b\mathbf{i} + d\mathbf{j})v$. The parallelogram R is the set of all vectors (9) with $(u,v) \in R^*$. Recall that the area is given by the magnitude of the crossproduct of the vectors in space representing adjacent sides: $\mathbf{A} = (a\mathbf{i} + c\mathbf{j})\triangle u + 0\mathbf{k}$ and $\mathbf{B} = (b\mathbf{i} + d\mathbf{j})\triangle u + 0\mathbf{k}$:

$$\text{(10)} \qquad \mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a\triangle u & c\triangle u & 0 \\ b\triangle v & d\triangle v & 0 \end{vmatrix} = \mathbf{k} \begin{vmatrix} a\triangle u & c\triangle u \\ b\triangle v & d\triangle v \end{vmatrix} = \ldots = \mathbf{k} \begin{vmatrix} a & b \\ c & d \end{vmatrix} \triangle u \triangle v$$

Hence we have proven that for a linear map (6)

$$\text{(11)} \qquad \text{Area}\,(R) = \left| \begin{vmatrix} a & b \\ c & d \end{vmatrix} \right| \triangle u \triangle v = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \text{Area}\,(R^*)$$

We remark that when the determinant (7) vanishes then the area of R vanishes. The map $(u,v) \to (x,y)$ given by (6) is not invertible if the determinant (7) vanishes. If the determinant vanishes then for each (x,y) there are either no solutions (u,v) to (6) or infinitely many. In fact, then the two vectors $\mathbf{A}$ and $\mathbf{B}$ above are parallel and the image R is just a line segment so we can only solve (6) if (x,y) is on this line and in then there are infinitely many solutions since each point on the line can be written in many ways as $u\mathbf{A} + v\mathbf{B}$, if $\mathbf{A}$ and $\mathbf{B}$ are parallel. That the map (6) is invertible if the determinant (7) is non-vanishing is the special case of the Inverse Function Theorem for linear maps. The inverse function theorem says that a map T is invertible close to a point if the derivative map DT is invertible at the point.

Th If T is a linear map with matrix A, $D = T(D^*)$ then Area$(D) = \det A$ Area(D^*). In fact, we can divide up region D^* into small rectangles and the area is the sum of the areas of these. This proves the change of variable theorem for linear maps. All maps can be approximated by a linear map close to a point. In the limit as the size of the rectangle R^* tends to 0 the ratio of the areas is given by (5).

Ex Let $D^* = \{(u,v);\ 0 \le u \le 1,\ 0 \le v \le 1\}$. Find $D = T(D^*)$ and verify the theorem above for the linear maps with matrices $A = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

Pf A just scales the x-axis so $D = \{(x,y);\ 0 \le x \le a,\ 0 \le y \le 1\}$, Area$(D) = a = \det A$. B is a rotation: $(1,0)$ rotates positively with an angle θ to $(\cos\theta, \sin\theta)$ and $(0,1)$ rotates to $(-\sin\theta, \cos\theta)$. Area$(D) =$ Area$(D^*) = 1$ and $\det B = 1$.