Lecture 13:. Let $T(u,v) = (x,y)$ be a mapping $x = x(u,v)$, $y = y(u,v)$, from a piecewise smooth simply connected domain D^* in the u-v plane onto a simply connected domain $D = T(D^*)$ in the x-y plane. Suppose that the **Jacobian determinant**:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} \neq 0$$

is non-vanishing so that the map is one-to-one, see the **Inverse Function Theorem** in section 3.5. Then the **change of variable theorem** states that

$$\iint_D f(x,y)\,dxdy = \iint_{D^*} f(x(u,v),y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dudv.$$

We proved this for a linear map if $f=1$ when it says that the area of D is the area of D^* times the Jacobian determinant which is the determinant of the linear map. The general case follows by dividing up D^* into smaller sets on which we can approximate the map by its linearization. If (u,v) is close to (u_i,v_j) then $\mathbf{T}(u,v)$ is by Taylor's theorem **approximated by the linear function**

$$\mathbf{T}(u,v) \sim \mathbf{T}(u_i,v_j) + \mathbf{T}_u(u_i,v_j)(u-u_i) + \mathbf{T}_v(u_i,v_j)(v-v_j)$$

where

$$\mathbf{T} = x\mathbf{i} + y\mathbf{j} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \mathbf{T}_u = \frac{\partial \mathbf{T}}{\partial u} = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} = \begin{bmatrix} \dfrac{\partial x}{\partial u} \\ \dfrac{\partial x}{\partial v} \end{bmatrix}, \quad \mathbf{T}_v = \frac{\partial \mathbf{T}}{\partial v} = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} = \begin{bmatrix} \dfrac{\partial x}{\partial u} \\ \dfrac{\partial x}{\partial v} \end{bmatrix}$$

The Jacobian is non-vanishing if and only $\mathbf{T}_u$ and $\mathbf{T}_v$ are not parallel.
We now **divide up the domain** D^* into smaller rectangles;

$$R_{ij}^* = \{(u,v);\ u_i \le u \le u_i + \triangle u,\ v_j \le v \le v_j + \triangle v\}$$

The image of R_{ij}^* under the map $\mathbf{T}: (u,v) \to (x,y)$ is a small set R_{ij} approximately equal to the image under the linear map. The image under the linear map is a parallelogram with adjacent sides given by vectors $\mathbf{T}_u \triangle u$ and $\mathbf{T}_v \triangle v$. The area of the parallelogram is the magnitude of the crossproduct of the vectors in space:

$$|\mathbf{T}_u \triangle u \times \mathbf{T}_v \triangle v| = \left| \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & 0 \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & 0 \end{vmatrix} \right| \triangle u \triangle v = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \triangle u \triangle v$$

and hence

$$\triangle A_{ij} = \text{Area}\,(R_{ij}) \sim \Big| \frac{\partial(x,y)}{\partial(u,v)}(u_i,v_j) \Big| \triangle u \triangle v$$

Now, with $(x_{ij},y_{ij}) = \big(x(u_i,v_j), y(u_i,v_j)\big) \in R_{ij}$ we have

$$\iint_{D^*} f(x(u,v),y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dudv$$

$$= \lim_{n\to\infty} \sum_{i,j=1}^{n} f\big(x(u_i,v_j), y(u_i,v_j)\big) \left| \frac{\partial(x,y)}{\partial(u,v)}(u_i,v_j) \right| \triangle u \triangle v$$

$$= \lim_{n\to\infty} \sum_{i,j=1}^{n} f\big(x_{ij},y_{ij}\big) \triangle A_{ij} = \iint_D f(x,y)\,dxdy$$

since its a **Riemann sum** for the double integral.

In **polar coordinates** $x = x(r,\theta) = r\cos\theta$, $y = y(r,\theta) = r\sin\theta$ we get

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \partial x/\partial r & \partial x/\partial\theta \\ \partial y/\partial r & \partial y/\partial\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\big(\cos^2\theta + \sin^2\theta\big) = r$$

and hence $\iint_D f(x,y)\,dxdy = \iint_{D^*} f(r\cos\theta, r\sin\theta)\,rdr\,d\theta$

Ex. Find the volume of the unit ball $B = \{(x,y,z);\ x^2+y^2+z^2 \le 1\}$.

Sol. $$\iiint_B 1\,dxdydz = \iint_{x^2+y^2\le 1} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} dz\ dxdy = \int_{x^2+y^2\le 1} 2\sqrt{1-x^2-y^2}\,dxdy$$
$$= \int_0^{2\pi}\int_0^1 2\sqrt{1-r^2}\ r\,dr\,d\theta = -\int_0^{2\pi} \frac{2}{3}(1-r^2)^{3/2}\Big|_0^1 d\theta = \int_0^{2\pi}\frac{2}{3} = \frac{4\pi}{3}.$$

Changing variables in Triple Integrals. If $\mathbf{T}(u,v,w) = (x,y,z)$ is an invertible mapping $x = x(u,v,w)$, $y = y(u,v,w)$, $z = z(u,v,w)$ and $W = T(W^*)$ then

$$\iiint_W f(x,y,z)\,dxdydz = \iiint_{W^*} f\big(x(u,v,w), y(u,v,w), z(u,v,w)\big)\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dudvdw,$$

where the Jacobian determinant is

$$\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}$$

Symbolically:

$$dxdydz = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dudvdw$$

In **spherical coordinates** $x = r\sin\phi\cos\theta$, $y = r\sin\phi\sin\theta$, $z = r\cos\phi$, $r \ge 0$, $0 \le \phi \le \pi$, $0 \le \theta < 2\pi$, we have $dxdydz = \dfrac{\partial(x,y,z)}{\partial(r,\phi,\theta)} dr\,d\phi\,d\theta = \cdots = r^2\sin\phi\,dr\,d\phi\,d\theta$

Ex. Find the volume of the unit ball $B = \{(x,y,z);\ x^2+y^2+z^2 \le 1\}$.

Sol.1 $$\iiint_B 1\,dxdydz = \int_0^{2\pi}\int_0^{\pi}\int_0^1 r^2\sin\phi\,dr\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi}\frac{r^3}{3}\Big|_0^1 \sin\phi\,d\phi\,d\theta$$
$$= \int_0^{2\pi}\int_0^{\pi}\frac{\sin\phi}{3}\,d\phi\,d\theta = \int_0^{2\pi} -\frac{\cos\phi}{3}\Big|_0^{\pi} d\theta \int_0^{2\pi}\frac{2}{3}\,d\theta = \frac{4\pi}{3}.$$

If $(x,y) = T(u,v)$ is an invertible mapping $(u,v) = T^{-1}(x,y)$ then

$$\frac{\partial(u,v)}{\partial(x,y)} = \left(\frac{\partial(x,y)}{\partial(u,v)}\right)^{-1}$$

i.e. the change of area going from (x,y) to (u,v) times the change going back is one.

Ex. Find $\iint_D xydxdy$, where $D = \{(x,y);\ 0 \le 2x-y \le 2,\ 0 \le y-x \le 1\}$.

Sol. Under the change of variables $u = 2x-y$ and $v = y-x$ the region becomes $D^* = \{(u,v)\ 0 \le u \le 2,\ 0 \le v \le 1\}$ and $x = u+v$, $y = 2u+v$. We have

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\frac{\partial(u,v)}{\partial(x,y)}\right|^{-1} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix}^{-1} = \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix}^{-1} = 1$$

so $\iint_D xy\,dxdy = \iint_{D^*}(u+v)(2u+v)\left|\frac{\partial(x,y)}{\partial(u,v)}\right|dudv = \int_0^2\int_0^1 (u+v)(2u+v)\,dudv = \ldots = 9.$