Lecture 15: Line Integrals.

Ex 1 Evaluate $\int_{\mathbf{c}_1} \mathbf{F}\cdot d\mathbf{s}$ and $\int_{\mathbf{c}_2} \mathbf{F}\cdot d\mathbf{s}$, where $\mathbf{F} = y\mathbf{i} + x\mathbf{j}$
$\mathbf{c}_1(t) = (1-t)\mathbf{i}$, $0 \le t \le 1$ and $\mathbf{c}_1(t) = (t-1)\mathbf{i} + (t-1)\mathbf{j}$, when $1 \le t \le 1 + 1/\sqrt{2}$ and $\mathbf{c}_2(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$, $0 \le t \le \pi/4$. Conclusion?
Sol.: We divide $\mathbf{c}_1$ up into two parts. When $0 \le t \le 1$ then $x = (1-t)$ and $y = 0$ so $dx/dt = -1$ and $dy/dt = 0$ and when $1 \le t \le 1 + 1/\sqrt{2}$ we have $x = (t-1)$, $y = (t-1)$ and $dx/dt = dy/dt = 1$ so

$$\int_{\mathbf{c}_1} \mathbf{F}\cdot d\mathbf{s} = \int_0^1 \Big(y\frac{dx}{dt} + x\frac{dy}{dt}\Big)\,dt + \int_1^{1/\sqrt{2}} \Big(y\frac{dx}{dt} + x\frac{dy}{dt}\Big)\,dt$$
$$= \int_0^1 0\,dt + \int_1^{1/\sqrt{2}} \big((t-1)1 + (t-1)\big)\,dt = (t-1)^2\Big|_1^{1+1/\sqrt{2}} = \frac{1}{2}$$

On $\mathbf{c}_2$ we have $x = \cos t$, $y = \sin t$, $dx/dt = -\sin t$, $dy/dt = \cos t$ so

$$\int_{\mathbf{c}_2} \mathbf{F}\cdot d\mathbf{s} = \int_0^{\pi/4} \big(y\frac{dx}{dt} + x\frac{dy}{dt}\big)\,dt = \int_0^{\pi/4} (-\sin^2 t + \cos^2 t)\,dt = \int_0^{\pi/4} \cos(2t)\,dt = \frac{\sin(2t)}{2}\Big|_0^{\pi/4} = \frac{1}{2}$$

Both line integrals go from $(0,1)$ to $(1/\sqrt{2}, 1/\sqrt{2})$ over different paths. In this case the value of the line integral is independent of the path. Vector fields for which this is true are called conservative and we shall study these more later. In physical terms the work is independent of the way for conservative vector fields.

Th(Line integrals of gradient vector fields) If $\mathbf{c} : [a,b] \to \mathbf{R}^3$ then

$$\int_{\mathbf{c}} \nabla f \cdot d\mathbf{s} = f(\mathbf{c}(b)) - f(\mathbf{c}(a))$$

Pf In fact, by the definition and chain rule

$$\int_{\mathbf{c}} \nabla f \cdot d\mathbf{s} = \int_a^b \nabla f(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\,dt = \int_a^b \frac{d}{dt} f(\mathbf{c}(t))\,dt = f(\mathbf{c}(b)) - f(\mathbf{c}(a))$$

A vector field $\mathbf{F}$ is called **conservative** if it has a **potential** f such that $\mathbf{F} = \nabla f$.

Ex. Let $\mathbf{F} = y\mathbf{i} + x\mathbf{j}$. Find a potential function and evaluate the line integrals in Ex 1 using the potential function in the theorem
Sol. We want to find f so that $\partial f/\partial x = y$, $\partial f/\partial y = x$ and $\partial f/\partial z = 0$. Integration of $\partial f/\partial x = y$ gives $f = xy + g(y,z)$, where g is any function of y and z. With this f it follows that $\partial f/\partial y = x$ and $\partial f/\partial z = 0$ if $g(y,z) = C$ is a constant. Hence $f = xy + C$, for any constant C, satisfies **grad** $f = \mathbf{F}$. By the theorem

$$\int_{\mathbf{c}_1} \mathbf{F}\cdot d\mathbf{s} = \int_{\mathbf{c}_2} \mathbf{F}\cdot d\mathbf{s} = f(1/\sqrt{2}, 1/\sqrt{2}, 0) - f(1,0,0) = \frac{1}{2}$$

Ex. Show that $\mathbf{F} = xy^2\mathbf{i} + x^3y\mathbf{j}$ is not conservative. **Sol.** If $\partial f/\partial x = xy^2$ then $\partial^2 f/\partial y\partial x = 2xy$ but if $\partial f/\partial y = x^3y$ then $\partial^2 f/\partial x\partial y = 3x^2y$ which is a contradiction.

A **Parameterized surface** is given in terms of two parameters

$$x = x(u,v), \quad y = y(u,v), \quad z = z(u,v), \qquad \text{or} \qquad \mathbf{T}(u,v) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$$

A particular example of a parameterized surface is a graph:

$$z = f(x,y), \qquad \text{or} \qquad \mathbf{T}(x,y) = x\mathbf{i} + y\mathbf{j} + f(x,y)\mathbf{k}$$

Ex The sphere $x^2+y^2+z^2=r^2$ can be parameterized using spherical coordinates:

$$x = r\sin\phi\cos\theta, \quad x = r\sin\phi\sin\theta, \quad z = r\cos\phi, \qquad 0\le\theta<2\pi, \quad 0\le\phi\le\pi$$

It can however, not be written as one graph, but one for the southern hemisphere $z=-\sqrt{r^2-x^2-y^2}$ and one for the northern hemisphere $z=\sqrt{r^2-x^2-y^2}$.

A surface is locally close to its tangent plane which is determined by its normal that we now will find. Another description of a surface is a level surface

$$h(x,y,z) = 0, \qquad \text{if} \qquad \nabla h(x,y,z) \ne \mathbf{0}.$$

(The graph is a special case with $h(x,y,z)=z-f(x,y)$.) In this case a normal is

$$\mathbf{N} = \nabla h$$

In particular the usual way to describe a plane is as a level surface:

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0,$$

where $\mathbf{N} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is a normal to the plane and $\mathbf{c}_0 = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$ is a point in the plane. However we can also give the **parametric equations of a plane**

$$\mathbf{T}(u,v) = \mathbf{A}u + \mathbf{B}v + \mathbf{c}_0,$$

where $\mathbf{A}$ and $\mathbf{B}$ are vectors in the plane. To go from the parametric equations to the usual equation we note that the normal to the plane is given by $\mathbf{N} = \mathbf{A}\times\mathbf{B}$.

To find the unit normal to a parameterized surface recall that for a parameterized curve we found the tangent by differentiating with respect to the parameter. Here, $u \to \mathbf{T}(u, v_j)$, where v_j is kept constant and u vary, is a parameterized curve and

$$\mathbf{T}_u = \frac{\partial \mathbf{T}}{\partial u} = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k}$$

is tangent to this curve, and hence to the surface. Similarly the vector

$$\mathbf{T}_v = \frac{\partial \mathbf{T}}{\partial v} = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}$$

is tangent to the curves $v \to \mathbf{T}(u_i, v)$, and hence to the surface. The tangent plane to the surface is spanned by $\mathbf{T}_u$ and $\mathbf{T}_v$ so a normal to the surface is given by

$$\mathbf{N} = \mathbf{T}_u \times \mathbf{T}_v$$

Ex 2 Find a normal to the surface $x = u\cos v$, $y = u\sin v$, $z = u$.
Sol $\mathbf{T} = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + u\,\mathbf{k}$ so $\mathbf{T}_u = \cos v\,\mathbf{i} + \sin v\,\mathbf{j} + \mathbf{k}$, $\mathbf{T}_v = -u\sin v\,\mathbf{i} + u\cos v\,\mathbf{j}$.
$\mathbf{N} = \mathbf{T}_u \times \mathbf{T}_v = \cdots = -u\cos v\,\mathbf{i} - u\sin v\,\mathbf{j} + u\,\mathbf{k}$.