

Lecture 16: Area of a parameterized surface.

A **Parameterized surface** is given in terms of two parameters

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v), \quad \text{or} \quad \mathbf{T}(u, v) = x \mathbf{i} + y \mathbf{j} + z \mathbf{k}$$

Now $u \rightarrow T(u, v_j)$ and $v \rightarrow T(u_i, v)$ are curves on the surface and hence

$$\mathbf{T}_u = \frac{\partial \mathbf{T}}{\partial u} = \frac{\partial x}{\partial u} \mathbf{i} + \frac{\partial y}{\partial u} \mathbf{j} + \frac{\partial z}{\partial u} \mathbf{k}, \quad \text{and} \quad \mathbf{T}_v = \frac{\partial \mathbf{T}}{\partial v} = \frac{\partial x}{\partial v} \mathbf{i} + \frac{\partial y}{\partial v} \mathbf{j} + \frac{\partial z}{\partial v} \mathbf{k}.$$

are **tangent vectors** to the surface. For $(u, v) \sim (u_i, v_j)$ the surface is close to its tangent plane at the point (u_i, v_j) and \mathbf{T} is close to the **linear approximation**:

$$\mathbf{T}(u, v) \sim \mathbf{L}_{ij}(u, v) = \mathbf{T}(u_i, v_j) + \mathbf{T}_u(u_i, v_j)(u - u_i) + \mathbf{T}_v(u_i, v_j)(v - v_j)$$

The image $S_{ij} = \mathbf{T}(R_{ij}^*)$ of a small rectangle

$$R_{ij}^* = \{(u, v); u_i \leq u \leq u_i + \Delta u, v_j \leq v \leq v_j + \Delta v\}.$$

under the map \mathbf{T} is close to the image under the linear map \mathbf{L}_{ij} . The image of R_{ij}^* under \mathbf{L}_{ij} is a parallelogram with adjacent sides $\mathbf{T}_u \Delta u$ and $\mathbf{T}_v \Delta v$ so

$$\text{Area}(S_{ij}) \sim \|\mathbf{T}_u \times \mathbf{T}_v\| \Delta u \Delta v = \|\mathbf{T}_u \times \mathbf{T}_v\| \text{Area}(R_{ij}^*)$$

Summing up over all small rectangles in the u - v plane we get

$$\text{Area}(S) = \sum \text{Area}(S_{ij}) \sim \sum \|\mathbf{T}_u \times \mathbf{T}_v(u_i, v_j)\| \Delta u \Delta v$$

and in the limit as $\Delta u, \Delta v \rightarrow 0$ we get the formula for the **surface area** of a parameterized surface:

$$\text{Area}(S) = \iint \|\mathbf{T}_u \times \mathbf{T}_v(u, v)\| \, du \, dv$$

Ex. Find the area of the sphere S of radius r .

Sol. Using the parametrization $\mathbf{T} = r \sin \phi \cos \theta \mathbf{i} + r \sin \phi \sin \theta \mathbf{j} + r \cos \phi \mathbf{k}$ we get $\mathbf{T}_\phi = r \cos \phi \cos \theta \mathbf{i} + r \cos \phi \sin \theta \mathbf{j} - r \sin \phi \mathbf{k}$ and $\mathbf{T}_\theta = -r \sin \phi \sin \theta \mathbf{i} + r \sin \phi \cos \theta \mathbf{j}$;

$$\begin{aligned} \mathbf{T}_\phi \times \mathbf{T}_\theta &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r \cos \phi \cos \theta & r \cos \phi \sin \theta & -r \sin \phi \\ -r \sin \phi \sin \theta & r \sin \phi \cos \theta & 0 \end{vmatrix} \\ &= r^2 \sin^2 \phi \cos \theta \mathbf{i} + r^2 \sin^2 \phi \sin \theta \mathbf{j} + r^2 \sin \phi \cos \phi \mathbf{k} \end{aligned}$$

and $|\mathbf{T}_\phi \times \mathbf{T}_\theta| = r^2 |\sin \phi| \sqrt{\sin^2 \phi \cos^2 \theta + \sin^2 \phi \sin^2 \theta + \cos^2 \phi} = r^2 \sin \phi$. Hence

$$\text{Area}(S) = \int_0^{2\pi} \int_0^\pi r^2 \sin \phi \, d\phi \, d\theta = \int_0^{2\pi} -r^2 \cos \phi \Big|_0^\pi \, d\theta = \int_0^{2\pi} 2r^2 \, d\theta = 4\pi r^2.$$

Surface area of a graph. In the special case of a graph $z = f(x, y)$ we have $\mathbf{T}(x, y) = x \mathbf{i} + y \mathbf{j} + f(x, y) \mathbf{k}$ we have $\mathbf{T}_x = \mathbf{i} + f_x(x, y) \mathbf{k}$, $\mathbf{T}_y = \mathbf{j} + f_y(x, y) \mathbf{k}$ and

$$\mathbf{T}_x \times \mathbf{T}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = -f_x \mathbf{i} - f_y \mathbf{j} + \mathbf{k}$$

and

$$\|\mathbf{T}_x \times \mathbf{T}_y\| = \sqrt{1 + f_x^2 + f_y^2}$$

and hence we get the formula for the area of a graph:

$$\text{Area}(S) = \iint \sqrt{1 + f_x^2 + f_y^2} \, dxdy$$

There is however a simpler way to remember this formula. Let

$$R_{ij}^* = \{(x, y); x_i \leq x \leq x_i + \Delta x, y_j \leq y \leq y_j + \Delta y\},$$

be a small rectangle in the x - y plane. Above this rectangle is a parallelogram in the tangent plane to the surface at $(x_i, y_j, f(x_i, y_j))$, that projects down to the rectangle in the x - y plane. The quotient of the area of the rectangle in the x - y plane to the area of the parallelogram in the tangent plane above it is the cosine of the angle γ between the tangent plane and the x - y plane. Hence

$$\text{Area}(S) = \iint \frac{dxdy}{|\cos \gamma|}$$

If \mathbf{n} is the unit normal to the tangent plane and \mathbf{k} is the normal to the x - y plane then the angle is given by $\cos \gamma = \mathbf{n} \cdot \mathbf{k}$. The unit normal to a graph $z = f(x, y)$ is easiest calculated by writing it in the form $h(x, y, z) = z - f(x, y) = 0$:

$$\mathbf{n} = \frac{\nabla h}{\|\nabla h\|} = \frac{-f_x \mathbf{i} - f_y \mathbf{j} + \mathbf{k}}{\sqrt{1 + f_x^2 + f_y^2}}$$

If we also take the inner product with \mathbf{k} the desired formula follows since

$$|\cos \gamma| = |\mathbf{n} \cdot \mathbf{k}| = \frac{1}{\sqrt{1 + f_x^2 + f_y^2}}$$

Ex Find the area of the part of the cone $S = \{(x, y, z); z = \sqrt{x^2 + y^2}, x^2 + y^2 \leq 1\}$.

Sol The surface is a graph and the angle between the surface and the x - y plane is $\gamma = \pi/4$, since when say $y = 0$ its just $z = |x|$. Hence $\cos \gamma = 1/\sqrt{2}$. Alternatively one can calculate $|\mathbf{n} \cdot \mathbf{k}|$. Hence

$$\text{Area}(S) = \int_{x^2+y^2 \leq 1} \frac{dxdy}{|\cos \gamma|} = \sqrt{2} \int_{x^2+y^2 \leq 1} dxdy = \sqrt{2} \times \text{Area of unit disc} = \sqrt{2}\pi.$$