

Lecture 17: Suppose we want to find the total volume of water in the oceans of the earth. At each point of the surface the depth of the ocean is given by a function f . To measure the total volume we divide up the surface S into smaller surface areas ΔS_{ij} , each of which is so small that we can think of it as approximately flat under which the depth of the ocean is approximately constant. The volume of water below ΔS_{ij} is approximately $f(x_{ij}, y_{ij}, z_{ij})\Delta S_{ij}$, where (x_{ij}, y_{ij}, z_{ij}) is any point in ΔS_{ij} . The total volume of water in the ocean is approximately $\sum_{i,j} f(x_{ij}, y_{ij}, z_{ij})\Delta S_{ij}$. We therefore define the **surface integral** of a function f over the surface S to be

$$\iint_S f \, dS = \lim_{\Delta S_{ij} \rightarrow 0} \sum_{i,j} f(x_{ij}, y_{ij}, z_{ij})\Delta S_{ij}$$

where the sum is over a partition of S into smaller surface areas ΔS_{ij} , (x_{ij}, y_{ij}, z_{ij}) is any point in ΔS_{ij} and we take the limit as the partition becomes finer.

Suppose that S is a parameterized surface: $\mathbf{T}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$, where $(u, v) \in D$. Let $R_{ij} = \{(u, v); u_i \leq u \leq u_i + \Delta u, v_j \leq v \leq v_j + \Delta v\}$ be a small rectangle in the u - v plane and let S_{ij} be the image of R_{ij} under the map $(u, v) \rightarrow \mathbf{T}(u, v)$. Then the area of S_{ij} is approximately the area in of the parallelogram in the tangent plane spanned by the vectors $\mathbf{T}_u \Delta u$ and $\mathbf{T}_v \Delta v$:

$$\Delta S_{ij} \sim \|\mathbf{T}_u \times \mathbf{T}_v(u_i, v_j)\| \Delta u \Delta v$$

Substituting this we get a Riemann sum for a double integral in the u - v plane. We therefore define the surface integral of a function f over a surface S :

$$\iint_S f \, dS = \iint_D f(x(u, v), y(u, v), z(u, v)) \|\mathbf{T}_u \times \mathbf{T}_v(u, v)\| \, du \, dv$$

We can symbolically write

$$dS = \|\mathbf{T}_u \times \mathbf{T}_v(u, v)\| \, du \, dv$$

Ex. Find $\iint_S z^2 \, dS$, where $S = \{(x, y, z); x^2 + y^2 + z^2 = 1\}$ is the unit sphere.

Sol. 1 A parametrization is $\mathbf{T}(\phi, \theta) = \sin \phi \cos \theta \mathbf{i} + \sin \phi \sin \theta \mathbf{j} + \cos \phi \mathbf{k}$, $0 \leq \phi \leq \pi$, $0 \leq \theta < 2\pi$, and we showed before that

$$dS = \|\mathbf{T}_\phi \times \mathbf{T}_\theta(\phi, \theta)\| \, d\phi \, d\theta = \sin \phi \, d\phi \, d\theta$$

Hence

$$\iint_S z^2 \, dS = \int_0^{2\pi} \int_0^\pi \cos^2 \phi \sin \phi \, d\phi \, d\theta = \int_0^{2\pi} \left. -\frac{\cos^3 \phi}{3} \right|_0^\pi \, d\theta = \int_0^{2\pi} \frac{2}{3} \, d\theta = \frac{4\pi}{3}$$

Sol. 2 A parametrization of the northern hemisphere S_+ is $\mathbf{T}(x, y) = x\mathbf{i} + y\mathbf{j} + f(x, y)\mathbf{k}$, where $f(x, y) = \sqrt{1 - x^2 - y^2}$ and

$$dS = \|\mathbf{T}_x \times \mathbf{T}_y\| \, dx \, dy = \sqrt{1 + f_x^2 + f_y^2} \, dx \, dy = \frac{dx \, dy}{\sqrt{1 - x^2 - y^2}}$$

Since the integral of z^2 over the southern and northern hemispheres are the same

$$\begin{aligned}\int_S z^2 dS &= 2 \int_{S_+} z^2 dS = 2 \iint_{x^2+y^2 \leq 1} (1-x^2-y^2) \frac{dxdy}{\sqrt{1-x^2-y^2}} \\ &= 2 \int_0^{2\pi} \int_0^1 (1-r^2)^{1/2} r dr d\theta = 2 \int_0^{2\pi} \frac{1}{3} (1-r^2)^{3/2} \Big|_0^1 d\theta = 2 \int_0^{2\pi} \frac{1}{3} d\theta = \frac{4\pi}{3}\end{aligned}$$

Ex. Find $\iint_S x dS$, where S is the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$.

Sol. The surface is a piece of a plane $ax+by+cz=d$ and putting in the 3 points we get $a=d$, $b=d$ and $c=d$, e.g. $a=b=c=d=1$. The surface is therefore given by $h(x, y, z) = x + y + z = 1$ and $(x, y) \in D = \{(x, y); x \geq 0, y \geq 0, x + y \leq 1\}$. The normal to the surface is therefore $\mathbf{n} = \nabla h / |\nabla h| = (\mathbf{i} + \mathbf{j} + \mathbf{k}) / \sqrt{3}$. We have

$$dS = \frac{dxdy}{|\cos \gamma|} = \frac{dxdy}{|\mathbf{n} \cdot \mathbf{k}|} = \sqrt{3} dxdy$$

If we rewrite $D = \{(x, y); 0 \leq x \leq 1, 0 \leq y \leq 1 - x\}$ we get

$$\iint_S x dS = \iint_D x \sqrt{3} dxdy = \int_0^1 \int_0^{1-x} x \sqrt{3} dy dx = \int_0^1 xy \sqrt{3} \Big|_{y=0}^{1-x} dx = \int_0^1 x(1-x) \sqrt{3} dx = \frac{\sqrt{3}}{6}$$

A surface is called **closed** if it has no boundary.

The sphere is closed but the upper hemisphere is not since its boundary is the equator. A closed surface has an inside and an outside.

An **oriented surface** is a two-sided surface with one side specified as the **outside**. The **outward** normal points away from from the outside of the surface.

A orientation can hence be specified by giving an outward normal everywhere.

The upper hemisphere is orientable.

A Möbius strip is not orientable since it only has one side.

A surface is called **regular** if it has a tangent plane everywhere. A parameterized surface is regular if $\mathbf{T}_u \times \mathbf{T}_v$ is nonvanishing everywhere.

A sphere is a regular surface but a cone is not regular at the tip.

There are a couple of alternative ways to express the surface area element:

$$dS = \|\mathbf{T}_u \times \mathbf{T}_v\| dudv = \sqrt{\|\mathbf{T}_u\|^2 \|\mathbf{T}_v\|^2 - (\mathbf{T}_u \cdot \mathbf{T}_v)^2} dudv$$

and since $\mathbf{T} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ we can also write (after some work)

$$dS = \|\mathbf{T}_u \times \mathbf{T}_v\| dudv = \sqrt{\left(\frac{\partial(x, y)}{\partial(u, v)}\right)^2 + \left(\frac{\partial(y, z)}{\partial(u, v)}\right)^2 + \left(\frac{\partial(x, z)}{\partial(u, v)}\right)^2} dudv$$

For a surface of revolution of a function $y = f(x)$ about the x -axis we have

$$\text{Area} = 2\pi \int_a^b |f(x)| \sqrt{1 + (f'(x))^2} dx$$

and revolved about the y axis

$$\text{Area} = 2\pi \int_a^b |x| \sqrt{1 + (f'(x))^2} dx$$