Lecture 18: Surface Integrals of vector functions.
The flow rate of fluid out of the total surface S, or the **flux** of the velocity vector field $\mathbf{F}$ out of the surface S, with outward unit normal $\mathbf{n}$, is given by

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$$

That only the normal component of $\mathbf{F}$ matters is clear since a tangential velocity would not contribute to the flow of fluid out from the surface.
The question of how to calculate the flux reduces how to calculate surface integrals. In a parametrization $\mathbf{T}=\mathbf{T}(u,v)$ we have

$$dS = \|\mathbf{T}_u\times\mathbf{T}_v\|\,dudv, \qquad \mathbf{n}=\pm\mathbf{T}_u\times\mathbf{T}_v/\|\mathbf{T}_u\times\mathbf{T}_v\|.$$

Hence

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \pm\iint \mathbf{F}\cdot(\mathbf{T}_u\times\mathbf{T}_v)\,dudv$$

Here the sign is positive if $\mathbf{T}_u\times\mathbf{T}_v$ points out from the surface.
Ex. Find the flux of $\mathbf{F}=x\,\mathbf{i}+y\,\mathbf{j}-2z\,\mathbf{k}$ out of the surface S of the cube $C=\{(x,y,z);\ 0\le x\le 1,\ 0\le y\le 1,\ 0\le z\le 1\}$.
Sol. The Cube has six sides S_1 with $x=0$, S_2 with $x=1$, S_3 with $y=0$, S_4 with $y=1$, S_5 with $z=0$ and S_6 with $z=1$. On S_1, the outward normal is $-\mathbf{i}$ and $\mathbf{F}\cdot\mathbf{n}=(y\,\mathbf{j}-2z\,\mathbf{k})\cdot(-\mathbf{i})=0$, on S_2, $\mathbf{F}\cdot\mathbf{n}=(\mathbf{i}+y\,\mathbf{j}-2z\,\mathbf{k})\cdot\mathbf{i}=1$, on $\mathbf{S}_3$, $\mathbf{F}\cdot\mathbf{n}=0$, on S_4, $\mathbf{F}\cdot\mathbf{n}=1$, on S_5, $\mathbf{F}\cdot\mathbf{n}=0$, and on S_6, $\mathbf{F}\cdot\mathbf{n}=-2$. Since the area of each side is one it follows that

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_{S_1}\mathbf{F}\cdot\mathbf{n}\,dS+...+\iint_{S_6}\mathbf{F}\cdot\mathbf{n}\,dS = 0+1+0+1+0-2=0$$

Ex. Find the flux of the vector field $\mathbf{F}=x\,\mathbf{i}+y\,\mathbf{j}-2z\,\mathbf{k}$ out of the sphere $S=\{(x,y,z);\ x^2+y^2+z^2=1\}$.
Sol. The surface can be written $h(x,y,z)=x^2+y^2+z^2=1$. The outward unit normal to the unit sphere is $\mathbf{n}=\nabla h/|\nabla h|=(x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k})/\sqrt{x^2+y^2+z^2}=x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}$, when $x^2+y^2+z^2=1$. Therefore

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_S x^2+y^2-2z^2\,dS$$

There are several ways to proceed:
(1) In Spherical coordinates, $\mathbf{T}(\phi,\theta)=\sin\phi\cos\theta\,\mathbf{i}+\sin\phi\sin\theta\,\mathbf{j}+\cos\phi\,\mathbf{k}$. Then $dS=|\mathbf{T}_\phi\times\mathbf{T}_\theta|d\phi d\theta=\sin\phi\,d\phi d\theta$. Hence

$$\begin{aligned}\iint_S \mathbf{F}\cdot\mathbf{n}\,dS &= \int_0^{2\pi}\int_0^{\pi}\big(\sin^2\phi\cos^2\theta+\sin^2\phi\sin^2\theta-2\cos^2\phi\big)\sin\phi\,d\phi\,d\theta\\ &= \int_0^{2\pi}\int_0^{\pi}(\sin^2\phi-2\cos^2\phi)\sin\phi\,d\phi d\theta = \int_0^{2\pi}\int_0^{\pi}(1-3\cos^2\phi)\sin\phi\,d\phi d\theta\\ &= \int_0^{2\pi} -\cos\phi+\cos^3\phi\Big|_{\phi=0}^{\pi}\,d\theta = 0\end{aligned}$$

(2) The sphere S can be written as the union of the northern hemisphere S_+ and southern hemisphere S_+ and each of these can be viewed as a graph over the x-y plane $S_\pm = \{(x, y, z);\ z = \sqrt{1 - x^2 - y^2},\ (x, y) \in D\}$, where $D = \{(x, y);\ x^2 + y^2 \leq 1\}$. Since the integrand and the sphere are symmetric under changing z to $-z$ we have

$$\iint_S x^2 + y^2 - 2z^2\, dS = 2 \iint_{S_+} x^2 + y^2 - 2z^2\, dS$$

We can now instead write $dS = \dfrac{dxdy}{|\cos\gamma|} = \dfrac{dxdy}{|\mathbf{n}\cdot\mathbf{k}|} = \dfrac{dxdy}{z} = \dfrac{dxdy}{\sqrt{1 - x^2 - y^2}}$ so

$$\iint_{S_+} x^2 + y^2 - 2z^2 dS = \iint_D \big(3(x^2 + y^2) - 2\big) \frac{dxdy}{\sqrt{1 - x^2 - y^2}}$$

Introducing polar coordinates we get

$$\iint_D \frac{dxdy}{\sqrt{1 - x^2 - y^2}} = \int_0^{2\pi} \int_0^1 (3r^2 - 2)(1 - r^2)^{-1/2} r\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 -3(1-r^2)^{1/2} r + (1-r^2)^{-1/2}\, dr\, d\theta = \int_0^{2\pi} (1-r^2)^{3/2} - (1-r^2)^{1/2} \Big|_{r=0}^{1} d\theta = 0.$$

(3) Finally, one can also use symmetry to see that

$$\iint_S x^2 dS = \iint_S y^2\, dS = \iint_S z^2\, dS.$$