Lecture 2: 1.3. The **vector or cross product** of the two vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ is the vector

$$\mathbf{a} \times \mathbf{b} = (a_2\, b_3 - a_3 b_2)\mathbf{i} + (a_3 b_1 - a_1 b_3)\mathbf{j} + (a_1 b_2 - a_2\, b_1)\mathbf{k}.$$

The geometric interpretation is $\|\mathbf{a}\|\,\|\mathbf{b}\|\,\sin\theta\,\mathbf{n}$, where $\mathbf{n}$ is a unit vector, $\|\mathbf{n}\| = 1$, that is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ and pointing in the direction so that $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{n}$ form a positively oriented system.
Note that $\mathbf{a} \times \mathbf{b} = 0$ if and only if $\mathbf{a}$ and $\mathbf{b}$ are parallel.

To remember the definition of vector product we introduce so called **determinants**.
A **determinant of order 2** is defined by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \tag{2}$$

Its magnitude is the **area** of the parallelogram with vectors (a, b) and (c, d) as edges.
A **determinant of order 3** is defined by

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \tag{3}$$

Its magnitude is the **volume** of the parallelepiped with vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ and $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$ as edges.

The cross product (2) is

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k} \tag{4}$$

Because of the similarity with (3), to remember this we symbolically write

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \tag{5}$$

We skipped the example below and the equations of a plane.
Equations of planes. $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$,
where (x_0, y_0, z_0) is a point in the plane and $\mathbf{n} = (a, b, c)$ is normal to the plane.
In fact, if (x, y, z) is a point in the plane then the vector $(x - x_0, y - y_0, z - z_0)$ is in the plane so its perpendicular to the normal: $(x - x_0, y - y_0, z - z_0) \cdot (a, b, c) = 0$.

Ex Find the equation of a plane passing through $(1, 0, 0)$, $(2, 2, 3)$,$(0, 2, 4)$.
Sol $\mathbf{a} = (2, 2, 3) - (1, 0, 0) = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = (0, 2, 4) - (1, 0, 0) = -\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$ are parallel to the plane and a normal to the plane is given by

$$\mathbf{n} = \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ -1 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 2 & 4 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 3 \\ -1 & 4 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix} \mathbf{k}$$

$$= \big(2 \cdot 4 - 3 \cdot 2\big)\mathbf{i} - \big(1 \cdot 4 - 3 \cdot (-1)\big)\mathbf{j} + \big(1 \cdot 2 - 2 \cdot (-1)\big)\mathbf{k} = 2\mathbf{i} - 7\mathbf{j} + 4\mathbf{k}$$

Let $(x_0, y_0, z_0) = (1, 0, 0)$. Hence the equation of the plane is $2(x - 1) - 7y + 4z = 0$.
Note that $\mathbf{a} \cdot \mathbf{n} = \mathbf{b} \cdot \mathbf{n} = 0$ as it should be.
Parametric equations of a plane If (x_0, y_0, z_0) is a point in the plane and $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ are two vectors in the plane then any point in the plane is given by $(x, y, z) = (x_0, y_0, z_0) + (a_1, a_2, a_3)\, t + (b_1, b_2, b_3)\, s$, $-\infty < s, t < \infty$.

1.5 Matrix algebra. An $m \times n$ matrix $A = [a_{ij}]$ is a collection of $m \cdot n$ numbers

$$A = \begin{array}{cc} \begin{array}{c} j \text{ th } \textbf{column} \end{array} & \\ \begin{bmatrix} a_{11} & \ldots & a_{1j} & \ldots & a_{1n} \\ & & \cdot & & \\ & & \cdot & & \\ a_{i1} & \ldots & a_{ij} & \ldots & a_{in} \\ & & \cdot & & \\ & & \cdot & & \\ a_{m1} & \ldots & a_{mj} & \ldots & a_{mn} \end{bmatrix} & i \text{ th } \textbf{row} \end{array}$$

A special case are the $1 \times n$ matrices which are row vector $(x_1, ..., x_n)$ or the $n \times 1$ matrices which are column vectors $\begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ x_n \end{bmatrix}$. Another important case are the square $n \times n$ matrices, e.g. a 2×2 matrix $\begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix}$ or a 3×3 matrix $\begin{bmatrix} 1 & 2 & 7 \\ 5 & 4 & 1 \\ 3 & 9 & 2 \end{bmatrix}$.

The **matrix multiplication** of an $m \times n$ matrix $A = [a_{ij}]$ by an $n \times k$ matrix $B = [b_{ij}]$ is the $m \times k$ matrix $C = [c_{ij}]$ whose entries are given by $c_{ij} = a_{i1} b_{1j} + \cdots + a_{im} b_{mj}$. The c_{ij} entry is the dot product between the i th row of A and the j th column of B

$$\begin{array}{cc} & \begin{bmatrix} b_{1j} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ b_{nj} \end{bmatrix} \\ \begin{bmatrix} \\ a_{i1} & . & . & . & a_{in} \\ \\ \end{bmatrix} & \begin{bmatrix} \\ c_{ij} \\ \\ \end{bmatrix} \end{array}$$

Any $m \times n$ matrix A determines a **mapping** $\mathbf{R}^n \ni \mathbf{x} \to A\mathbf{x} \in \mathbf{R}^m$ defined by

$$A\mathbf{x} = \begin{bmatrix} a_{11} & \ldots & a_{1n} \\ & \cdot & \\ & \cdot & \\ a_{m1} & \ldots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + & \ldots & + a_{1n}x_n \\ & \cdot & \\ & \cdot & \\ a_{m1}x_1 + & \ldots & + a_{mn}x_n \end{bmatrix}.$$

The map is seen to be linear:

$$A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y}, \qquad A(\alpha \mathbf{x}) = \alpha A\mathbf{x}$$

Conversely, a linear map is exactly one given by matrix multiplication.

If B is an $n \times k$ matrix then multiplication first by B and then A

$$\mathbf{x} \xrightarrow{\text{multiply by } B} B\mathbf{x} \xrightarrow{\text{multiply by } A} A(B\mathbf{x})$$

defines a map $\mathbf{R}^k \ni \mathbf{x} \to A(B\mathbf{x}) \in \mathbf{R}^m$. The matrix product AB is constructed so that multiplying by the matrix AB

$$\mathbf{x} \xrightarrow{\qquad \text{multiply by } AB \qquad} (AB)\mathbf{x}$$

is the same as first multiplying by B and then by A, i.e. $(AB)\mathbf{x} = A(B\mathbf{x})$.

Let us conclude the discussion by some examples.

Ex 1 $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \to \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$ **rotates** vectors an angle $\pi/2$ counterclockwise.

Ex 2 $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \to \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 \\ 3x_2 \end{bmatrix}$ **scales** vectors by a factor 3.

Ex 3 $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \to \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3x_2 \\ 3x_1 \end{bmatrix}$ scales and rotates vectors.