Lecture 20: Green's theorem. Suppose that D is a domain in the plane with boundary curve C going in positive direction, i.e. walking in the direction of C the domain D should be on your left. **Green's theorem** says that

$$\int_C P dx + Q dy = \iint_D \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \Big) dxdy$$

This somewhat similar to the first fundamental theorem of calculus:

$$f(b) - f(a) = \int_a^b f'(x)\, dx.$$

If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ then the left of Green's theorem is $\int_C \mathbf{F}\cdot d\mathbf{s}$, and $\mathbf{curl}\, \mathbf{F} = \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \Big)\mathbf{k}$

Ex Let $P = xy$, $Q = y^2$ and $D = \{(x, y);\ 0 \le x \le 1,\ x^2 \le y \le x\}$. Let C be the positively oriented boundary of D. Calculate both sides of Green's theorem.
Sol. The boundary consists of two parts C_1; $x = t$, $y = t^2$, $0 \le t \le 1$ and C_2; $x = 1-t$, $y = 1-t$, $0 \le t \le 1$. Note that C_2 is oriented so it starts at $(x, y) = (1, 1)$ in order that the total curve should be positively oriented. Let $P = xy$ and $Q = y^2$.

$$\int_C P\, dx + Q\, dy = \int_{C_1} \Big(xy\frac{dx}{dt} + y^2 \frac{dy}{dt} \Big)\, dt + \int_{C_2} \Big(xy\frac{dx}{dt} + y^2 \frac{dy}{dt} \Big)\, dt$$

$$= \int_0^1 t^3 + 2t^5\, dt + \int_0^1 -(1-t)^2 - (1-t)^2\, dt = \frac{t^4}{4} + \frac{t^6}{3}\Big|_0^1 + \frac{2(1-t)^3}{3}\Big|_0^1 = \frac{1}{4} + \frac{1}{3} - \frac{2}{3} = -\frac{1}{12}.$$

On the other hand

$$\iint_D \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \Big)\, dxdy = \iint_D -x\, dxdy = \int_0^1 \int_{x^2}^x -x\, dy\, dx = \int_0^1 -x(x - x^2)\, dx$$

$$= -\frac{x^3}{3} + \frac{x^4}{4}\Big|_0^1 = -\frac{1}{3} + \frac{1}{4} = -\frac{1}{12}.$$

From Green's theorem applied to $(P, Q) = (0, x)$ and $(P, q) = (-y, 0)$ we get

$$\text{Area}\,(D) = \iint_D 1\, dxdy = \int_C x\, dy = -\int_C y\, dx = \frac{1}{2}\int_C -y\, dx + x\, dy$$

which gives another way to calculate the area.

Ex. Find the area of the interior of an ellipse: $D = \{(x, y);\ (x/a)^2 + (y/b)^2 \le 1\}$
Sol. Parameterizing the ellipse $x = a\cos t$, $y = b \sin t$, $0 \le t \le 2\pi$ and using (5.4.2)

$$\text{Area}\,(D) = \frac{1}{2}\int_C -y\, dx + x\, dy = \frac{1}{2}\int_0^{2\pi} -y\frac{dx}{dt} + x\frac{dy}{dt}\, dt$$

$$= \frac{1}{2}\int_0^{2\pi} -b\sin t(-a \sin t) + (a\cos t) b \sin t\, dt = \frac{1}{2}\int_0^{2\pi} ab(\sin^2 t + \cos^2 t)\, dt = \pi ab$$

Proof of Green's theorem. The proof reduces to proving the two equalities

$$\int_C P dx = \iint_D -\frac{\partial P}{\partial y} dxdy, \qquad \text{and} \qquad \int_C Q dy = \iint_D \frac{\partial Q}{\partial x} dxdy,$$

since the theorem has to be true separately for the cases $(P, 0)$ and $(0, Q)$. Let us just prove the first equality for P since the proof for the second is similar.

Suppose D is a region of type I: $D = \{(x, y);\ a \le x \le b,\ f_1(x) \le y \le f_2(x)\}$. By the Fundamental Theorem of Calculus

$$-\iint_D \frac{\partial P}{\partial y} dxdy = -\int_a^b \int_{f_1(x)}^{f_2(x)} \frac{\partial P}{\partial y} dy\, dx = \int_a^b -P(x, f_2(x))\, dx + \int_a^b P(x, f_1(x))\, dx$$

The boundary C of D consists of four parts, a bottom C_1 where $y = f_1(x)$, a right side C_2 where $x = b$, a top C_3 where $y = f_2(x)$, a left side C_4 where $x = a$. Each of these four curves have to be positively oriented. At the top C_2 we have $x = t$, $y = f_2(t)$, $a \le t \le b$:

$$\int_{C_1} P\, dx = \int_a^b P(t, f_1(t))\, \frac{dx}{dt} dt = \int_a^b P(x, f_1(x))\, dx$$

On the right side we have $x = b$, $y = t$, $f_1(b) \le t \le f_2(t)$, so $\frac{dx}{dt} = 0$ so $\int_{C_2} P dx = 0$. The top C_3 is oriented to go backwards from $x = b$ to $x = 2a$. The integral over C_3 is therefor $-$ the integral over the curve C_3^- going in the reverse direction $x = t$, $y = f_2(t)$, $a \le t \le b$:

$$\int_{C_3} P\, dx = -\int_{C_3^-} P\, dx = -\int_a^b P(t, f_2(t))\, \frac{dx}{dt} dt = -\int_a^b P(x, f_2(x))\, dx$$

The integral over C_4 also vanishes as above. Adding up the contributions from the different curves proves the theorem for P in case D is a region of type I.

If D is not a region of type I we can divide D up into regions D_1,...,D_N of type I. The right hand side integral over D in Green's Theorem is then just sum of the integrals over the regions D_1,...,D_N. Similarly the integral over C is the sum of the integrals over their boundaries C_1,...,C_N.