Lecture 21: Stokes' theorem. Let S be a surface with unit normal $\mathbf{n}$ and positively oriented boundary C, i.e. if you walk in the direction of the curve on the side of the normal then the surface should be on your left. **Stokes' theorem** says

$$\int_C \mathbf{F}\cdot d\mathbf{s} = \iint_S \mathbf{curl}\,\mathbf{F}\cdot\mathbf{n}\,dS$$

if $\mathbf{F}$ is a smooth vector field on S.
If S is a domain in the x-y plane then Stoke's theorem reduces to Green's theorem. In fact $\int_C \mathbf{F}\cdot d\mathbf{s} = \int_C P dx + Q dy$, if $\mathbf{F} = P\mathbf{i}+Q\mathbf{j}$ and $\mathbf{curl}\,\mathbf{F}\cdot\mathbf{n} = \dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}$, if $\mathbf{n}=\mathbf{k}$.
Ex. Find the integral $\int_C -y^3 dx + x^3 dy - z^3 dz$, where C is the intersection of the cylinder $x^2+y^2=1$ and the plane $x+y+z=1$ and the orientation of C corresponds to a counterclockwise motion in the x-y plane.
Sol. 1. Let $\mathbf{F} = -y^3\mathbf{i} + x^3\mathbf{j} - z^3\mathbf{k}$. The integral is by Stokes Theorem equal to the surface integral of $\mathbf{curl}\,\mathbf{F}\cdot\mathbf{n}$ over some surface S with the boundary C and with unit normal positively oriented with respect to the orientation of the boundary. We have $\mathbf{curl}\,\mathbf{F} = ... = (3x^2+3y^2)\mathbf{k}$. We take S to be the region in the plane $h(x,y,z) = x+y+z = 1$ with boundary C. A unit normal to S is given by $\mathbf{n} = \nabla h/|\nabla h| = (\mathbf{i}+\mathbf{j}+\mathbf{k})/\sqrt{3}$ and it has the correct orientation since $\mathbf{n}\cdot\mathbf{k} = 1/\sqrt{3} > 0$. We therefore get

$$\int_C \mathbf{F}\cdot d\mathbf{s} = \iint_S \mathbf{curl}\,\mathbf{F}\cdot\mathbf{n}\,dS = \iint_S 3(x^2+y^2)/\sqrt{3}\,dS$$

Writing $dS = dxdy/|\mathbf{n}\cdot\mathbf{k}| = \sqrt{3}dxdy$ we get

$$\iint_{x^2+y^2\le 1} 3(x^2+y^2)\,dxdy = \int_0^{2\pi}\int_0^1 3r^2\,rdr\,d\theta = \int_0^{2\pi} \frac{3}{4}r^4\Big|_0^1\,d\theta = 2\pi\frac{3}{4} = \frac{3\pi}{2}$$

Sol. 2. Directly calculating the line integral. Parameterizing the curve C we can write $x=\cos t$, $y=\sin t$ and $z = 1-x-y = 1-\cos t-\sin t$, $0\le t\le 2\pi$ and write

$$\int_C -y^3dx + x^3dy - z^3dz = \int_0^{2\pi}\Big(-y^3\frac{dx}{dt} + x^3\frac{dy}{dt} - z^3\frac{dz}{dt}\Big)\,dt$$
$$= \int_0^{2\pi}\big(\sin^4 t + \cos^4 t + (1-\cos t-\sin t)^3(\sin t-\cos t)\,dt$$

But this is to much work to calculate.

Furthermore, Stokes Theorem can alternatively be used to define the curl: The component of $\mathbf{curl}\,\mathbf{F}$ in the direction of a unit vector $\mathbf{n}$ is defined to be the limit as $\varepsilon\to 0$ of the line integral of $\mathbf{F}$ around a small circle C_ε of radius ε perpendicular to $\mathbf{n}$, divided by the area of the disc S_ε enclosed by C_ε:

$$\int_{C_\varepsilon} \mathbf{F}\cdot d\mathbf{s} = \iint_{S_\varepsilon} \mathbf{curl}\,\mathbf{F}\cdot\mathbf{n}\,dS = \mathbf{curl}\,\mathbf{F}\cdot\mathbf{n}\ \ \mathrm{Area}(S_\varepsilon)$$

where $\mathbf{curl}\,\mathbf{F}\cdot\mathbf{n}$ is evaluated at some point on S_ε. It follows that

$$\mathbf{curl}\,\mathbf{F}\cdot\mathbf{n} = \lim_{\varepsilon\to 0}\frac{\int_{C_\varepsilon}\mathbf{F}\cdot d\mathbf{s}}{\mathrm{Area}(S_\varepsilon)}$$

Ex. Show that $\int_C ye^z dx + xe^z dy + xye^z dz = 0$ for a closed curve C.
Sol. $\mathbf{F} = \nabla(xye^z)$ so $\mathbf{curl}\,\mathbf{F} = 0$ and by Stokes's theorem the integral vanishes.
Ex. Find $\int_{C_a} \mathbf{F}\cdot d\mathbf{s}$, where $\mathbf{F} = (-y\mathbf{i}+x\mathbf{j})/(x^2+y^2)$ and C_a is the circle $x^2+y^2 = a^2$ in the x-y plane going counterclockwise.
Sol. $\mathbf{curl}\,\mathbf{F} = ... = 0$. Hence one would have thought that by Stokes theorem the line integral would vanish. However, if we parameterize $x = a\cos t$ and $y = a\sin t$,$0 \le t < 2\pi$, we get

$$\int_{C_a} \mathbf{F}\cdot d\mathbf{s} = \int_0^{2\pi} \Big(\frac{-y}{x^2+y^2}\frac{dx}{dt} + \frac{x}{x^2+y^2}\frac{dy}{dt}\Big)\,dt$$
$$= \int_0^{2\pi} \Big(\frac{-a\sin t(-a\sin t)}{a^2} + \frac{a\cos t(a\cos t)}{a^2}\Big)\,dt = \int_0^{2\pi} \sin^2 t + \cos^2 t\,dt = 2\pi$$

The reason Stokes' theorem failed to hold in this case was that the vector field $\mathbf{F}$ is singular when $(x,y) = (0,0)$, i.e. along the z-axis.

Proof of Stokes' theorem. for a graph $z = f(x,y)$, $(x,y) \in D$. Since the surface can be written $h(x,y,z) = z - f(x,y)$ a normal is given by $\mathbf{N} = \nabla h = -f_x\mathbf{i} - f_y\mathbf{j} + \mathbf{k}$ and the unit normal is given by $\mathbf{n} = \mathbf{N}/|\mathbf{N}|$. The surface measure is $dS = dxdy/|\mathbf{k}\cdot\mathbf{n}|$, where $\mathbf{k}\cdot\mathbf{n} = \mathbf{k}\cdot\mathbf{N}/|\mathbf{N}| = 1/|\mathbf{N}|$, so $dS = |\mathbf{N}|\,dxdy$ and hence

$$\iint_S \mathbf{G}\cdot\mathbf{n}\,dS = \iint_D -G_1 f_x - G_2 f_y + G_3\,dxdy, \qquad \text{if} \quad \mathbf{G} = G_1\mathbf{i} + G_2\mathbf{j} + G_3\mathbf{k}$$

If we apply to $\mathbf{F}$ this to $\mathbf{G} = \mathbf{curl}\,\mathbf{F}$ we get

$$\iint_S \mathbf{curl}\,\mathbf{F}\cdot\mathbf{n}\,dS = \iint_D -\Big(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\Big)f_x - \Big(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\Big)f_y + \Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big)\,dxdy,$$

If we parameterize the boundary $x = x(t)$, $y = y(t)$ and $z = f(x,y)$ we have

$$\frac{dz}{dt} = f_x\frac{dx}{dt} + f_y\frac{dy}{dt},$$

by the chain rule, and

$$\int_C \mathbf{F}\cdot d\mathbf{s} = \int_a^b \Big(F_1\frac{dx}{dt} + F_2\frac{dy}{dt} + F_3\frac{dz}{dt}\Big)\,dt = \int_a^b \Big(\big(F_1 + f_x F_3\big)\frac{dx}{dt} + \big(F_2 + f_y F_3\big)\frac{dy}{dt}\Big)\,dt$$

This can now be considered as a line integral in the plane:

$$\int_C \mathbf{F}\cdot d\mathbf{s} = \int_{\partial D} P dx + Q\,dy, \qquad \text{where}$$

$$P(x,y) = F_1(x,y,f(x,y)) + f_x(x,y)F_3(x,y,f(x,y)),$$
$$Q(x,y) = F_2(x,y,f(x,y)) + f_y(x,y)F_3(x,y,f(x,y))$$

We can therefore apply Greens formula in the plane.

$$\frac{\partial P}{\partial y} = \frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}f_y + f_x\frac{\partial F_3}{\partial y} + f_x f_y\frac{\partial F_3}{\partial z} + f_{xy}F_3$$
$$\frac{\partial Q}{\partial x} = \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial z}f_x + f_y\frac{\partial F_3}{\partial x} + f_x f_y\frac{\partial F_3}{\partial z} + f_{xy}F_3$$

so by Green's theorem

$$\int_C \mathbf{F}\cdot d\mathbf{s} = \int_{\partial D} P dx + Q\,dy = \iint_D \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\,dxdy$$
$$= \iint_D -\Big(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\Big)f_x - \Big(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\Big)f_y + \Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big)\,dxdy,$$