Lecture 22: Conservative Fields.

A vector field is called **gradient** if it is a gradient $\mathbf{F} = \mathbf{grad}\ \phi$ of a scalar **potential**.

It is called **path independent** if the line integral depends only on the endpoints, i.e. if $\mathbf{c}_1$ and $\mathbf{c}_2$ are any two paths from P to Q then $\int_{\mathbf{c}_1} \mathbf{F}\cdot d\mathbf{s} = \int_{\mathbf{c}_2} \mathbf{F}\cdot d\mathbf{s}$. This is equivalent to that the line integral along any closed path or loop vanishes.

Th A vector field $\mathbf{F}$ in a domain D is gradient if and only if it is path independent. In that case we say that it is **conservative** and the integral is the difference in potential of the endpoint minus initial point: $\int_{\mathbf{c}} \mathbf{F}\cdot d\mathbf{s} = \phi(Q) - \phi(P)$.

Ex. Evaluate $\int_{\mathbf{c}} \mathbf{F}\cdot d\mathbf{s}$, where $\mathbf{F} = y\mathbf{i} + x\mathbf{j}$ and $\mathbf{c} = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$, $0 \le t \le \pi/4$.
Sol. We want to find ϕ so that $\partial\phi/\partial x = y$, $\partial\phi/\partial y = x$ and $\partial\phi/\partial z = 0$. Integration of $\partial\phi/\partial x = y$ gives $\phi = xy + g(y,z)$, where g is any function of y and z. With this ϕ it follows that $\partial\phi/\partial y = x$ and $\partial\phi/\partial z = 0$ if $g(y,z) = C$ is a constant. Hence $\phi = xy + C$, for any constant C, satisfies $\mathbf{grad}\ \phi = \mathbf{F}$. Hence

$$\int_c \mathbf{F}\cdot d\mathbf{s} = \phi(1/\sqrt{2}, 1/\sqrt{2}, 0) - \phi(1,0,0) = \frac{1}{2}$$

Note that $\mathbf{F} = \mathbf{grad}\ \phi$ is perpendicular to the level surfaces of ϕ and hence the level surfaces of the potential function are perpendicular to the flow lines of $\mathbf{F}$.

Proof of Th: If $\mathbf{F} = \mathbf{grad}\ \phi$ then for any curve $\mathbf{c}$ from P to Q:

$$\int_{\mathbf{c}} \mathbf{F}\cdot d\mathbf{s} = \int_a^b \frac{\partial\phi}{\partial x}\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\frac{dy}{dt} + \frac{\partial\phi}{\partial z}\frac{dz}{dt}\,dt = \int_a^b \frac{d}{dt}\phi(x(t), y(t), z(t))\,dt = \phi(Q) - \phi(P),$$

If the integral is independent of the way then we define

$$\phi(x,y,z) = \int_{(x_0,y_0,z_0)}^{(x,y,z)} \mathbf{F}\cdot d\mathbf{s},$$

where the integral is along any curve from a fixed point (x_0, y_0, z_0) to (x,y,z). This is well-defined since it does not depend on which path we integrate along. In particular we can pick a curve going from (x_0, y_0, z_0) to (a, y, z) and then along a straight line segment (t, y, z), $a \le t \le x$, to (x, y, z):

$$\phi(x,y,z) = \int_{(x_0,y_0,z_0)}^{(a,y,z)} \mathbf{F}\cdot d\mathbf{s} + \int_{(a,y,z)}^{(x,y,z)} \mathbf{F}\cdot d\mathbf{s} = \int_{(x_0,y_0,z_0)}^{(a,y,z)} \mathbf{F}\cdot d\mathbf{s} + \int_a^x F_1(t,y,z)dt.$$

The first integral is independent of x and by the fundamental theorem of calculus

$$\frac{\partial\phi}{\partial x}(x,y,z) = \frac{\partial}{\partial x}\int_a^x F_1(t,y,z)dt = F_1(x,y,z).$$

The proof of that $\partial\phi/\partial y = F_2$ and $\partial\phi/\partial z = F_3$ is similar.

Ex. Show that $\mathbf{F} = xy^2\mathbf{i} + x^3y\mathbf{j}$ is not conservative. **Sol.** If $\partial\phi/\partial x = xy^2$ then $\partial^2\phi/\partial y\partial x = 2xy$ but if $\partial\phi/\partial y = x^3y$ then $\partial^2\phi/\partial x\partial y = 3x^2y$ which is a contradiction.

Conservative fields-Irrotational fields. We have just seen an example of a vector field $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ that could not be conservative because if there was a potential $\mathbf{F} = \mathbf{grad}\,\phi$ then $\partial^2\phi/\partial x\partial y = \partial^2\phi/\partial y\partial x$ etc. so we must have $\partial F_1/\partial y = \partial F_2/\partial x$ etc. Hence for a vector field to be conservative we must have

$$\mathbf{curl}\,\mathbf{F} = \Big(\frac{\partial F_2}{\partial z} - \frac{\partial F_3}{\partial y}\Big)\mathbf{i} + \Big(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\Big)\mathbf{j} + \Big(\frac{\partial F_1}{\partial y} - \frac{\partial F_2}{\partial x}\Big)\mathbf{k} = \mathbf{0}$$

A vector field satisfying this is called **irrotational**. We have

Theorem. *A vector field* $\mathbf{F}$ *defined and continuously differentiable throughout a simply connected domain* D *is conservative if and only if it is irrotational in* D.

The physical interpretation of this is that the flow lines for a gradient vector field can not curl around in a closed orbit since ϕ increases in the direction of the gradient.
Ex. Show that $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ is conservative in all space.
Sol. By the previous theorem it suffices to show that it is irrotational: $\mathbf{curl}\,\mathbf{F} = \mathbf{0}$.
Ex. Is $\mathbf{F} = (-y\mathbf{i} + x\mathbf{j})/(x^2 + y^2)$ conservative in $D = \{(x, y, z);\ (x, y) \neq (0, 0)\}$.
Sol. $\nabla \times \mathbf{F} = ... = \mathbf{0}$ but D is not simply connected so it does not follow that it is conservative. In fact, it is not since the line line integral along a circle around the z-axis is nonvanishing as we shall see. If $x = \cos t$, $y = \sin t$ and $z = 0$, $0 \le t \le 2\pi$ then $\int_C \mathbf{F} \cdot d\mathbf{s} = \int_0^{2\pi} (F_1 dx/dt + F_2 dy/dt + F_3 dz/dt)dt = \int_0^{2\pi} \sin^2 t + \cos^2 t\, dt = 2\pi$.
Ex. Show that the vector field above in the domain D of space where $x > 0$ is conservative and find a potential there.
Sol. That it is conservative follows from that $\nabla \times \mathbf{F} = \mathbf{0}$ and that the domain D is simply connected. $\phi = \tan^{-1}(y/x)$ is a potential in D, since $x \neq 0$ in D.

Proof of the theorem. That conservative implies irrotational is just the calculation above that $\nabla \times \nabla\phi = 0$. We shall prove that irrotational implies conservative if the domain is all of space or a rectangular box containing the origin. We define

$$\phi(x, y, z) = \int_0^x F_1(t, 0, 0)\, dt + \int_0^y F_2(x, t, 0)\, dt + \int_0^z F_3(x, y, t)\, dt$$

and we will show that $\nabla\phi = \mathbf{F}$ if $\nabla \times \mathbf{F} = 0$. By the Fundamental Theorem of Calculus

$$\frac{\partial \phi}{\partial z}(x, y, z) = \frac{d}{dz}\Big|_{(x,y)-fixed} \phi(x, y, z) = \frac{d}{dz} \int_0^z F_3(x, y, t)\, dt = F_3(x, y, z)$$

Similarly, also using that we can differentiate below an integral sign:

$$\frac{\partial \phi}{\partial y}(x, y, z) = F_2(x, y, 0) + \int_0^z \frac{\partial F_3}{\partial y}(x, y, t)\, dt$$

If we now also use the fact that $\partial F_3/\partial y = \partial F_2/\partial z$ we obtain

$$\frac{\partial \phi}{\partial y}(x, y, z) = F_2(x, y, 0) + \int_0^z \frac{d}{dt}\Big|_{(x,y)-fixed} F_2(x, y, t)\, dt$$
$$= F_2(x, y, 0) + F_2(x, y, z) - F_2(x, y, 0) = F_2(x, y, z)$$

by the Fundamental Theorem of Calculus. Finally by the same argument

$$\frac{\partial \phi}{\partial x}(x, y, z) = F_1(x, 0, 0) + \int_0^y \frac{\partial F_2}{\partial x}(x, t, 0)\, dt + \int_0^z \frac{\partial F_3}{\partial x}(x, y, t)\, dt$$
$$= F_1(x, 0, 0) + \int_0^y \frac{d}{dt} F_1(x, t, 0)\, dt + \int_0^z \frac{d}{dt} F_1(x, y, t)\, dt$$
$$= F_1(x, 0, 0) + F_1(x, y, 0) - F_1(x, 0, 0) + F_1(x, y, z) - F_1(x, y, 0) = F_1(x, y, z)$$