

Lecture 6: 3.2 Taylor's formula in several variables.

Recall Taylor's formula for $f: \mathbf{R} \rightarrow \mathbf{R}$:

$$(1) \quad f(x) = f(a) + f'(a)(x-a) + \frac{f''}{2}(a)(x-a)^2 + \dots + \frac{f^{(k)}(a)}{k!}(x-a)^k + R_k(x, a)$$

where the remainder or error tends to 0 faster than the previous terms when $x \rightarrow a$:

$$(2) \quad |R_k(x, a)| \leq \frac{M}{(k+1)!}|x-a|^{k+1}, \quad \text{if } |f^{(k+1)}(z)| \leq M,$$

for $|z-a| \leq |x-a|$. The Taylor polynomial $P_k = f_k - R_k$ is the polynomial of degree k that best approximates $f(x)$ for x close to a . It is chosen so its derivatives of order $\leq k$ are equal to the derivatives of f at a . (2) follows from repeated integration of

$$(2b) \quad \frac{d^{k+1}}{dx^{k+1}} R_k(x, a) = f^{k+1}(x), \quad \left. \frac{d^j}{dx^j} R_k(x, a) \right|_{x=a} = 0, \quad j \leq k.$$

A similar formula holds for functions of several variables $\mathbf{F}: \mathbf{R}^n \rightarrow \mathbf{R}^m$. In order to state it we first write $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbf{R}^n$ and $\mathbf{a} = (a_1, \dots, a_n) \in \mathbf{R}^n$.

$$(3) \quad \mathbf{F}(\mathbf{x}) = \mathbf{F}(\mathbf{a}) + \sum_{i=1}^n \mathbf{F}_{x_i}(\mathbf{a})(x_i - a_i) + \frac{1}{2} \sum_{i,j=1}^n \mathbf{F}_{x_i x_j}(\mathbf{a})(x_i - a_i)(x_j - a_j) \\ + \dots + \frac{1}{k!} \sum_{i_1, \dots, i_k=1}^n \mathbf{F}_{x_{i_1} \dots x_{i_k}}(\mathbf{a})(x_{i_1} - a_{i_1}) \dots (x_{i_k} - a_{i_k}) + \mathbf{R}_k(\mathbf{a}, \mathbf{x}),$$

where the remainder or error tends to 0 faster than the previous terms when $x \rightarrow a$:

$$(4) \quad |\mathbf{R}_k(x, a)| \leq \frac{M}{(k+1)!} |\mathbf{x} - \mathbf{a}|^{k+1}, \quad \text{if } \sum_{i_1, \dots, i_{k+1}=1}^n |\mathbf{F}_{x_{i_1} \dots x_{i_{k+1}}}(\mathbf{z})| \leq M,$$

for $|\mathbf{z} - \mathbf{a}| \leq |\mathbf{x} - \mathbf{a}|$. Here

$$(5) \quad \mathbf{F}_{x_{i_1} \dots x_{i_k}} = \frac{\partial^k \mathbf{F}}{\partial x_{i_1} \dots \partial x_{i_k}}$$

First, the general case reduces to the case $m = 1$ by considering each component of $\mathbf{F} = (F_1, \dots, F_m)$ and we may hence assume that $\mathbf{F}: \mathbf{R}^n \rightarrow \mathbf{R}$. In order to prove (3) we introduce $\mathbf{x} - \mathbf{a} = \mathbf{h}$ and apply the one dimensional Taylor's formula (1) to the function $f(t) = \mathbf{F}(\mathbf{x}(t))$ along the line segment $\mathbf{x}(t) = \mathbf{a} + t\mathbf{h}$, $0 \leq t \leq 1$:

$$(6) \quad f(1) = f(0) + f'(0) + f''(0)/2 + \dots + f^{(k)}(0)/k! + R_k$$

Here $f(1) = \mathbf{F}(\mathbf{a} + \mathbf{h})$, i.e. the left hand side of (3), $f(0) = \mathbf{F}(\mathbf{a})$, i.e. the first term in the right hand side of (3), and by the chain rule

$$(7) \quad f'(t) = \frac{d}{dt} \mathbf{F}(\mathbf{x}(t)) = \sum_{i=1}^n \mathbf{F}_{x_i}(\mathbf{x}(t)) \frac{dx_i}{dt} = \sum_{i=1}^n \mathbf{F}_{x_i}(\mathbf{x}(t)) h_i$$

and hence $f'(0)$ is the second term in the right of (3). Repeating this gives

$$(8) \quad f''(t) = \frac{d}{dt} \sum_{i=1}^n \mathbf{F}_{x_i}(\mathbf{x}(t)) h_i = \sum_{i=1, j=1}^n \mathbf{F}_{x_i x_j}(\mathbf{x}(t)) h_i h_j$$

and this gives the third term and so on.

If $f: \mathbf{R}^2 \rightarrow \mathbf{R}$, $\mathbf{a} = (0, 0)$ and $\mathbf{x} = (x, y)$ then the second degree Taylor polynomial is

$$f(x, y) \sim f(0, 0) + f_x(0, 0)x + f_y(0, 0)y + \frac{1}{2}(f_{xx}(0, 0)x^2 + 2f_{xy}(0, 0)xy + f_{yy}(0, 0)y^2)$$

Ex. Let $f(x, y) = 3 + 2x + x^2 + 2xy + 3y^2 + x^3 - y^4$. Find the second degree Taylor polynomial around $\mathbf{a} = (0, 0)$.

Sol. The second degree Taylor polynomial is

$$\begin{aligned} f(0, 0) + f_x(0, 0)x + f_y(0, 0)y + \frac{1}{2}(f_{xx}(0, 0)x^2 + 2f_{xy}(0, 0)xy + f_{yy}(0, 0)y^2) \\ = 3 + 2x + \frac{1}{2}(2x^2 + 2 \cdot 2xy + 6y^2) = 3 + 2x + x^2 + 2xy + 3y^2 \end{aligned}$$

The derivative of a vector field as a linear map. Let $\mathbf{F}: \mathbf{R}^n \rightarrow \mathbf{R}^m$ be a vector field. Then we can think of the derivative of \mathbf{F} at the point $\mathbf{a} \in \mathbf{R}^n$ as the linear map $\mathbf{DF}: \mathbf{R}^n \rightarrow \mathbf{R}^m$, mapping the vector $\mathbf{h} = (h_1, \dots, h_n)$ to the vector

$$\mathbf{DF}(\mathbf{a})\mathbf{h} = \lim_{t \rightarrow 0} \frac{\mathbf{F}(\mathbf{a} + t\mathbf{h}) - \mathbf{F}(\mathbf{a})}{t} = \frac{\partial \mathbf{F}}{\partial x_1}(\mathbf{a})h_1 + \dots + \frac{\partial \mathbf{F}}{\partial x_n}(\mathbf{a})h_n,$$

This is the linear map that best approximates the function close to \mathbf{a} :

$$\mathbf{F}(\mathbf{a} + \mathbf{h}) = \mathbf{F}(\mathbf{a}) + \mathbf{DF}(\mathbf{a})\mathbf{h} + \mathbf{R}_2(\mathbf{a}, \mathbf{h}), \quad \text{where } |\mathbf{R}_2(\mathbf{a}, \mathbf{h})| \leq M|\mathbf{h}|^2$$

tends to 0 faster than the other terms as $|\mathbf{h}| \rightarrow 0$.

We didn't have time to cover this: **Ex.** Let $\mathbf{F}(x, y) = (1+x-y+x^2)\mathbf{i} + (x^2-y^2+y^4)\mathbf{k}$. Find the Taylor polynomial of degree one for $\mathbf{F}(x, y)$ around $(x, y) = (1, 0)$. **Sol.**

$$\frac{\partial \mathbf{F}}{\partial x}(x, y) = ((1+2x)\mathbf{i} + 2x\mathbf{j}), \quad \frac{\partial \mathbf{F}}{\partial y}(x, y) = (-\mathbf{i} + (-2y + 4y^3)\mathbf{j})$$

Hence

$$\mathbf{DF}(1, 0)(x-1, y) = \frac{\partial \mathbf{F}}{\partial x}(1, 0)(x-1) + \frac{\partial \mathbf{F}}{\partial y}(1, 0)y = (3\mathbf{i} + 2\mathbf{j})(x-1) - \mathbf{i}y$$

so

$$\begin{aligned} \mathbf{F}(x, y) &= \mathbf{F}(1, 0) + \mathbf{DF}(1, 0)(x-1, y) + \mathbf{R}_2(1, 0)(x-1, y) \\ &= 3\mathbf{i} + \mathbf{j} + (3\mathbf{i} + 2\mathbf{j})(x-1) - \mathbf{i}y + \mathbf{R}(1, 0)(x-1, y) \\ &= (3x-y)\mathbf{i} + (2x-1)\mathbf{j} - \mathbf{R}_2(1, 0)(x-1, y) \end{aligned}$$

In fact, the remainder

$$\begin{aligned} \mathbf{R}_2(1, 0)(x-1, y) &= (1+x-y+x^2)\mathbf{i} + (x^2-y^2+y^4)\mathbf{k} - \left((3x-y)\mathbf{i} + (2x-1)\mathbf{j} \right) \\ &= (1+x^2-2x)\mathbf{i} + (x^2-2x+1+y^2+y^4)\mathbf{j} = (x-1)^2\mathbf{i} + ((x-1)^2+y^2+y^4)\mathbf{j} \end{aligned}$$

is at least quadratically small in $(x-1)$ and y , i.e. it is bounded by a constant times $(x-1)^2 + y^2$, when $(x-1)^2 + y^2$ is small.