Lecture 7: 4.2 Arc length. (The material below only took half a lecture so in the future I might move it and do it together with path integrals in section 7.1).

If $\mathbf{c}$ is a vector valued function

$$\mathbf{c}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}, \qquad a \leq t \leq b$$

then we defined the **derivative** to be

$$\mathbf{c}'(t) = \lim_{\triangle t \to 0} \frac{\mathbf{c}(t+\triangle t) - \mathbf{c}(t)}{\triangle t} = x'(t)\,\mathbf{i} + y'(t)\,\mathbf{j} + z'(t)\,\mathbf{k}.$$

We define the **arc length** of the curve to be

$$L = \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2}\, dt$$

However, since

$$|\mathbf{c}'(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$$

this can also be written:

$$L = \int_a^b |\mathbf{c}'(t)|\, dt$$

It is easy to see that it is the arc length if we approximate it by a Riemann sum:

$$L \sim \sum_{i=0}^{n-1} |\,\mathbf{c}'(t_i)|\triangle t$$

where $t_i = a + i\triangle t$, $\triangle t = (b-a)/n$ and use the definition of derivative:

$$\mathbf{c}'(t_i) = \lim_{h \to 0} \frac{\mathbf{c}(t_i + h) - \mathbf{c}(t_i)}{h} \sim \frac{\mathbf{c}(t_i + \triangle t) - \mathbf{c}(t_i)}{\triangle t},$$

if $\triangle t$ is small, we obtain

$$L \sim \sum_{i=0}^{n-1} |\,\mathbf{c}(t_i + \triangle t) - \mathbf{c}(t_i)|.$$

This is exactly the length of the polygon consisting of the line segments between the vertices $\mathbf{c}(t_i)$, $i = 0, ..., n$, which is a good approximation of the arc length.

Ex. Find the arc length of the helix $\mathbf{c}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$, when $0 \leq t \leq 2\pi$.
Sol. We have $\mathbf{c}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \,\mathbf{k}$, so $|\mathbf{c}'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$.
Hence $L = \displaystyle\int_0^{2\pi} |\,\mathbf{c}'(t)|\, dt = \int_0^{2\pi} \sqrt{2}\, dt = \sqrt{2}\, 2\pi.$

The same curve can be represented by different **parameterizations**:
Ex. Find the arc length of the helix: $\mathbf{c}_2(u) = \cos u^2 \mathbf{i} + \sin u^2 \mathbf{j} + u^2 \mathbf{k}$, $0 \leq u \leq \sqrt{2\pi}$.
Sol. We have $\mathbf{c}_2'(u) = -2u \sin u^2\,\mathbf{i} + 2u\cos u^2\,\mathbf{j} + \,2u\mathbf{k}$. Hence
$|\mathbf{c}_2'(u)| = \sqrt{4u^2\sin^2 u^2 + 4u^2\cos^2 u^2 + 4u^2} = 2\sqrt{2}\,u$ and

$$L = \int_0^{\sqrt{2\pi}} |\mathbf{c}_2'(u)|\, du = \int_0^{\sqrt{2\pi}} 2\sqrt{2}u\, du = \begin{bmatrix} u^2 = t, \\ 2u\, du = dt \end{bmatrix} = \int_0^{2\pi} \sqrt{2}\, dt = 2\sqrt{2}\pi$$

Two different parameterizations, $\mathbf{c}(t) = \mathbf{c}_2(u)$, where $u = u(t)$, leads to the same arc length. By the chain rule $\mathbf{c}'(t) = d\mathbf{c}_2(u)/dt = \mathbf{c}_2'(u)u'(t)$ and if we change variables

$$\int |\,\mathbf{c}_2'(u)|\, du = \begin{bmatrix} u = u(t), \\ du = u'(t)\, dt \end{bmatrix} = \int |\,\mathbf{c}'(t)|\, dt.$$