Lecture 13: Section 15.6: The change of variable theorem. Suppose that $x(u)$, $a \le u \le b$ is a change of variables. In order for it to be invertible we assume that $x'(u) > 0$, when $a \le u \le b$. Then we can change variables in the integral:

$$\int_{x(a)}^{x(b)} f(x)\, dx = \int_a^b f(x(u)) \frac{dx}{du} du \tag{15.6.1}$$

We will give similar theorem for functions of two variables. Let $T(u,v) = (x,y)$;

$$x = x(u,y), \qquad y = y(u,v), \tag{15.6.2}$$

be a **mapping** from a piecewise smooth simply connected domain $\tilde{D}$ in the u-v plane **onto** a domain D in the x-y plane. We require that the map is smooth and in order for it to be invertible we assume that the **Jacobian determinant**:

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \tag{15.6.3}$$

is nonvanishing. We say that a map is **one-to-one** if no two points are mapped to one point, or formulated differently if $T(u,v) = T(u',v')$ implies that $(u,v) = (u',v')$.

Change of variable theorem.

$$\iint_D f(x,y)\, dxdy = \iint_{\tilde{D}} f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dudv \tag{15.6.4}$$

If $f = 1$ then the left is the area of the domain D. We will argue that the Jacobian gives the local change of area scale under the mapping, symbolically

$$dxdy = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dudv \tag{15.6.5}$$

For the purpose of giving a sense to this statement let us study a linear map:

$$\begin{cases} x = au + bv, \\ y = cu + dv, \end{cases} \tag{15.6.6}$$

for which the Jacobian determinant is a constant

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc. \tag{15.6.7}$$

Let $\tilde{R}$ be a small rectangle in the u-v plane with side lengths $\triangle u$ and $\triangle v$

$$\tilde{R} = \{(u,v);\ 0 \le u \le \triangle u,\ 0 \le v \le \triangle v\} \tag{15.6.8}$$

Then we claim that the image of this rectangle in the x-y plane R is a small parallelogram with adjacent sides formed by the two vectors $(a\mathbf{i} + c\mathbf{j})\triangle u$ and $(b\mathbf{i} + d\mathbf{j})\triangle v$. To see this we write (15.6.6) as vector equation in matrix notation

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} u + \begin{bmatrix} b \\ d \end{bmatrix} v, \qquad \text{where} \qquad \begin{bmatrix} x \\ y \end{bmatrix} = x\mathbf{i} + y\mathbf{j} \quad \text{etc.} \tag{15.6.9}$$

which is just another way of saying that $x\mathbf{i}+y\mathbf{j}=(a\mathbf{i}+c\mathbf{j})u+(b\mathbf{i}+d\mathbf{j})v$. The parallelogram R is the set of all vectors (15.6.9) with $(u,v)\in\tilde{R}$. Recall that the area is given by the magnitude of the crossproduct of the vectors in space representing adjacent sides: $\mathbf{A}=(a\mathbf{i}+c\mathbf{j})\triangle u+0\mathbf{k}$ and $\mathbf{B}=(b\mathbf{i}+d\mathbf{j})\triangle u+0\mathbf{k}$:

$$\mathbf{A}\times\mathbf{B}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ a\triangle u&c\triangle u&0\\ b\triangle v&d\triangle v&0\end{vmatrix}=\mathbf{k}\begin{vmatrix}a\triangle u&c\triangle u\\ b\triangle v&d\triangle v\end{vmatrix}=\ldots=\mathbf{k}\begin{vmatrix}a&b\\ c&d\end{vmatrix}\triangle u\triangle v \tag{15.6.10}$$

Hence we have proven that for a linear map (15.6.6)

$$\text{Area}\,(R)=\left|\begin{vmatrix}a&b\\ c&d\end{vmatrix}\right|\triangle u\triangle v=\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\text{Area}\,(\tilde{R}) \tag{15.6.11}$$

All maps can be approximated by a linear map close to a point and the meaning of (15.6.5) is that in the limit as the size of the rectangle $\tilde{R}$ tends to 0 the ratio of the areas is given by (15.6.5).

We remark that when the determinant (15.6.7) vanishes then the area of R vanishes even if the area of $\tilde{R}$ is nonvanishing. In this case the mapping $(u,v)\to(x,y)$ given by (15.6.6) is not invertible. If the determinant (15.6.7) vanishes then for each (x,y) there are either no solutions (u,v) to (15.6.6) or infinitely many solutions. In fact, in this case the two vectors $\mathbf{A}$ and $\mathbf{B}$ above are parallel and the image R is just a line segment so we can only solve (15.6.6) if (x,y) is on this line and in that case there are infinitely many solutions since each point on the line can be written in many ways as linear combinations of the two vectors; $u\mathbf{A}+v\mathbf{B}$.

Let us now show how to find the inverse of (15.6.6) when (15.6.7) is nonvanishing. For this it is useful to write (15.6.6) in matrix notation

$$\begin{cases}x=au+bv,\\ y=cu+dv,\end{cases}\qquad\Leftrightarrow\qquad\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}a&b\\ c&d\end{bmatrix}\begin{bmatrix}u\\ v\end{bmatrix}, \tag{15.6.12}$$

This is just to be interpreted as a way to write the equation (15.6.6), where the right hand side just means the right hand side of (15.6.9). It is however convenient to think of the collection of the four numbers in (a,b,c,d) determining the mapping (15.6.6) as an entity called a (2×2) matrix representing the mapping:

$$A=\begin{bmatrix}a&b\\ c&d\end{bmatrix}$$

The multiplication of the matrix with the vector in the right hand side of (15.6.12) is simply defined to be right hand side of (15.6.9). If the determinant is nonvanishing then the linear map is invertible and the inverse is another linear map that is represented by a matrix, i.e. a collection of four numbers (a',b',c',d') such that

$$\begin{cases}u=a'x+b'y,\\ v=c'x+d'y,\end{cases}\qquad\Leftrightarrow\qquad\begin{bmatrix}u\\ v\end{bmatrix}=\begin{bmatrix}a'&b'\\ c'&d'\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} \tag{15.6.13}$$

If we solve the system (15.6.12) for (u,v) in terms of (x,y) we will get an expression of the form (15.6.13) and it turns out that the matrix is

$$A^{-1}=\begin{bmatrix}a'&b'\\ c'&d'\end{bmatrix}=\frac{1}{ad-bc}\begin{bmatrix}d&-b\\ -c&a\end{bmatrix}=\begin{bmatrix}\dfrac{d}{ad-bc}&\dfrac{-b}{ad-bc}\\ \dfrac{-c}{ad-bc}&\dfrac{a}{ad-bc}\end{bmatrix}, \tag{15.6.14}$$

A calculation gives the determinant of the inverse as the inverse of the determinant:

$$\det A^{-1}=1/\det A \tag{15.6.15}$$