Section 4.6: Surface Area. An **Orientation** of a surface is a continuous choice of a positive direction of the unit normal or an outside of the surface. We say that a surface is **orientable** or two-sided if it has an orientation. A closed surface without a boundary, like the sphere, has such an orientation. However, the Möbius strip is not orientable, when you walk around it you can come back to the other side.

A **Parametrized surface** is given in terms of two parameters

$$(4.6.1) \qquad x = x(u,v), \quad y = y(u,v), \quad z = z(u,v), \qquad \text{or} \qquad \mathbf{R} = \mathbf{R}(u,v)$$

A particular example of a parameterized surface is a graph:

$$(4.6.2) \qquad z = f(x,y), \qquad \text{or} \qquad \mathbf{R} = x\mathbf{i} + y\mathbf{j} + f(x,y)\mathbf{k}$$

Ex. The sphere $x^2+y^2+z^2 = r^2$ can be parametrized using spherical coordinates:

$$(4.6.3) \quad x = r\sin\phi\cos\theta, \quad x = r\sin\phi\sin\theta, \quad z = r\cos\phi, \qquad 0 \le \theta < 2\pi, \quad 0 \le \phi \le \pi$$

It can however, not be written as one graph, but one for the southern hemisphere $z = -\sqrt{r^2 - x^2 - y^2}$ and one for the northern hemisphere $z = \sqrt{r^2 - x^2 - y^2}$.

A surface is locally close to its tangent plane which is determined by its normal that we know will find. Another description of a surface is a level surface

$$(4.6.4) \qquad h(x,y,z) = 0, \qquad \text{if} \qquad \nabla h(x,y,z) \neq \mathbf{0}.$$

((4.6.2) is a special case with $h(x,y,z) = z - f(x,y)$.) In this case a unit normal is

$$(4.6.7) \qquad \mathbf{n} = \nabla h / |\nabla h|$$

To find the unit normal to (4.6.1) we recall that for a parametrized curve we found the tangent by differentiating with respect to the parameter. Here, $\mathbf{R}(u, v_j)$, where v_j is kept constant and u vary, is a parametrized curve and the vector

$$(4.6.8) \qquad \mathbf{R}_u = \frac{\partial \mathbf{R}}{\partial u} = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k}$$

is tangent to this curve. Similarly the vector $\mathbf{R}_v = \partial\mathbf{R}/\partial v$ is tangent to the curves $\mathbf{R}(u_i, v)$, when u_i is constant and v varying. The tangent plane to the surface is spanned by the two vectors $\mathbf{R}_u$ and $\mathbf{R}_v$ and a normal to the surface is hence given by

$$(4.6.9) \qquad \mathbf{N} = \mathbf{R}_u \times \mathbf{R}_v$$

The actual surface at a point (u,v) is close to its tangent plane at the point (u_i, v_j):

$$(4.6.10) \qquad \mathbf{R}(u,v) \sim \mathbf{R}(u_i,v_j) + \mathbf{R}_u(u_i,v_j)(u-u_i) + \mathbf{R}(u_i,v_j)(v-v_j)$$

if (u,v) vary over a small rectangle

$$(4.6.11) \qquad \tilde{R}_{ij} = \{(u,v);\, u_i \le u \le u_i + \triangle u, , v_j \le v \le v_j + \triangle v\}.$$

The image of this rectangle under the map $(u,v) \to \mathbf{R}(u,v)$ is a small surface S_{ij}. The area of S_{ij} is approximately the area of the image of $\tilde{R}_{ij}$ under the linear map (4.6.10), which is the parallelogram with the adjacent sides $\mathbf{R}_u \triangle u$ and $\mathbf{R}_v \triangle v$ so

$$(4.6.12) \qquad \text{Area}\,(S_{ij}) \sim |\mathbf{R}_u \times \mathbf{R}_v(u_i,v_j)| \triangle u \triangle v = |\mathbf{R}_u \times \mathbf{R}_v| \text{Area}\,(\tilde{R}_{ij})$$

Summing up over all small rectangles in the u-v plane we get

$$(4.6.13) \qquad \text{Area}\,(S) = \sum \text{Area}\,(S_{ij}) \sim \sum |\mathbf{R}_u \times \mathbf{R}_v(u_i,v_j)| \triangle u \triangle v$$

and in the limit as $\triangle u, \triangle v \to 0$ we get

$$(4.6.14) \qquad \text{Area}\,(S) = \iint |\mathbf{R}_u \times \mathbf{R}_v(u,v)|\, du dv$$

In the special case of a graph $z = f(x,y)$ we have $\mathbf{R} = x\,\mathbf{i} + y\,\mathbf{j} + f(x,y)\,\mathbf{k}$ we have $\mathbf{R}_x = \mathbf{i} + f_x(x,y)\,\mathbf{k}$ and $\mathbf{R}_y = \mathbf{j} + f_y(x,y)\,\mathbf{k}$ and

$$\mathbf{R}_x \times \mathbf{R}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = -f_x\,\mathbf{i} - f_y\,\mathbf{j} + \mathbf{k} \tag{4.6.15}$$

and

$$|\mathbf{R}_x \times \mathbf{R}_y| = \sqrt{1 + f_x^2 + f_y^2} \tag{4.6.16}$$

and hence we get the formula for the area of a graph:

$$\text{Area}\,(S) = \iint \sqrt{1 + f_x^2 + f_y^2}\; dxdy \tag{4.6.17}$$

There is however a simpler way to remember this formula. Let

$$\tilde{R}_{ij} = \{(x,y);\, x_i \le x \le x_i + \triangle x,\, y_j \le y \le y_j + \triangle y\}, \tag{4.6.18}$$

be a small rectangle in the x-y plane. Above this rectangle is a parallelogram in the tangent plane to the surface at $(x_i, y_j, f(x_i, y_j))$, that projects down to the rectangle in the x-y plane. The quotient of the area of the rectangle in the x-y plane to the area of the parallelogram in the tangent plane above it is the cosine of the angle γ between the tangent plane and the x-y plane. Hence

$$\text{Area}\,(S) = \iint \frac{dxdy}{|\cos\gamma|} \tag{4.6.19}$$

If $\mathbf{n}$ is the unit normal to the tangent plane and $\mathbf{k}$ is the normal to the x-y plane then the angle is given by $\cos\gamma = \mathbf{n}\cdot\mathbf{k}$. The unit normal to a graph $z = f(x,y)$ is easiest calculated by writing it in the form $h(x,y,z) = z - f(x,y)$ and using (4.6.7):

$$\mathbf{n} = \frac{-f_x\,\mathbf{i} - f_y\,\mathbf{j} + \mathbf{k}}{\sqrt{1 + f_x^2 + f_y^2}} \tag{4.6.20}$$

If we also take the inner product with $\mathbf{k}$ we obtain:

$$|\cos\gamma| = |\mathbf{n}\cdot\mathbf{k}| = \frac{1}{\sqrt{1 + f_x^2 + f_y^2}} \tag{4.6.21}$$

and substituting this into (4.6.19) gives (4.6.17).

Ex. Find the area of the sphere S of radius r.

Sol. Using the parametrization $\mathbf{R} = r\sin\phi\cos\theta\,\mathbf{i} + r\sin\phi\sin\theta\,\mathbf{j} + r\cos\phi\mathbf{k}$ we get $\mathbf{R}_\phi = r\cos\phi\cos\theta\,\mathbf{i} + r\cos\phi\sin\theta\,\mathbf{j} - r\sin\phi\mathbf{k}$ and $\mathbf{R}_\theta = -r\sin\phi\sin\theta\mathbf{i} + r\sin\phi\cos\theta\mathbf{j}$;

$$\mathbf{R}_\phi \times \mathbf{R}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r\cos\phi\cos\theta & r\cos\phi\sin\theta & -r\sin\phi \\ -r\sin\phi\sin\theta & r\sin\phi\cos\theta & 0 \end{vmatrix} = r^2\sin^2\!\phi\cos\theta\,\mathbf{i} + r\sin^2\!\phi\sin\theta\,\mathbf{j} + r^2\sin\phi\cos\phi\,\mathbf{k}$$

and $|\mathbf{R}\phi \times \mathbf{R}_\theta| = r^2|\sin\phi|\sqrt{\sin^2\phi\cos^2\theta + \sin^2\phi\sin^2\theta + \cos^2\phi} = r^2\sin\phi$. Hence $\text{Area}\,(S) = \int_0^{2\pi}\int_0^{\pi} r^2\sin\phi\, d\phi d\theta = \int_0^{2\pi} -r^2\cos\phi\big|_0^{\pi}\, d\theta = \int_0^{2\pi} 2r^2\, d\theta = 4\pi r^2.$