Lecture 16: 4.7 Surface Integrals. We want to find the total volume of water in the ocean. At each point of the surface of the earth the depth of the ocean is given by a function f. For the purpose of measuring the total volume we divide up the surface of the earth S into smaller surface areas $\triangle S_{ij}$, each of which is so small that we can think of it as approximately a flat piece of a plane under which the depth of the ocean is approximately constant. Then the volume of water below $\triangle S_{ij}$ is approximately $f(x_{ij}, y_{ij}, z_{ij})\triangle S_{ij}$, where (x_{ij}, y_{ij}, z_{ij}) is any point in $\triangle S_{ij}$. Hence the total volume of water in the ocean is approximately $\sum_{i,j} f(x_{ij}, y_{ij}, z_{ij})\triangle S_{ij}$

We therefore think the **surface integral** of a function f over the surface S as

$$\iint_S f\, dS = \lim_{\triangle S_{ij}\to 0} \sum_{i,j} f(x_{ij}, y_{ij}, z_{ij})\triangle S_{ij} \tag{4.7.1}$$

where the sum is over a partition of S into smaller surface areas $\triangle S_{ij}$, (x_{ij}, y_{ij}, z_{ij}) is any point in $\triangle S_{ij}$ and we take the limit as the partition becomes finer.

Suppose that the surface S is parametrized by $\mathbf{R}(u,v) = x(u,v)\,\mathbf{i} + y(u,v)\,\mathbf{j} + z(u,v)\,\mathbf{k}$, where $(u,v)\in D$. Let $R_{ij} = \{(u,v);\ u_i \le u \le u_i+\triangle u,\ v_j \le v \le v_j+\triangle v\}$ be a small rectangle in the u-v plane and let S_{ij} be the image of R_{ij} under the map $(u,v)\to \mathbf{R}(u,v)$. Then the area of S_{ij} is approximately the area in of the parallelogram in the tangent plane spanned by the vectors $\mathbf{R}_u\triangle u$ and $\mathbf{R}_v\triangle v$:

$$\triangle S_{ij} \sim \big|\mathbf{R}_u \times \mathbf{R}_v(u_i, v_j)\big|\triangle u \triangle v \tag{4.7.2}$$

Substituting this into (4.7.1) we get a Riemann sum for a double integral in the u-v plane. We therefore define the surface integral of a function f over a surface S:

$$\iint_S f\, dS = \iint_D f\big(x(u,v), y(u,v), z(u,v)\big)\big|\mathbf{R}_u \times \mathbf{R}_v(u,v)\big|\, dudv \tag{4.7.3}$$

We can symbolically write

$$dS = \big|\mathbf{R}_u \times \mathbf{R}_v(u,v)\big|\, dudv \tag{4.7.4}$$

Ex. Find $\iint_S z^2\, dS$, where $S = \{(x,y,z);\ x^2+y^2+z^2 = 1\}$ is the unit sphere.

Sol. A parametrization is $\mathbf{R}(\phi,\theta) = \sin\phi\cos\theta\,\mathbf{i}+\sin\phi\sin\theta\,\mathbf{j}+\cos\phi\,\mathbf{k}$, $0\le\phi\le\pi$, $0\le\theta<2\pi$, and we showed before that

$$dS = \big|\mathbf{R}_\phi \times \mathbf{R}_\theta(\phi,\theta)\big|d\phi d\theta = \sin\phi d\phi d\theta \tag{4.7.5}$$

Hence

$$\iint_S z^2\, dS = \int_0^{2\pi}\!\!\int_0^{\pi} \cos^2\phi\, \sin\phi\, d\phi\, d\theta = \int_0^{2\pi} -\frac{\cos^3\phi}{3}\Big|_0^{\pi}\, d\theta = \int_0^{2\pi} \frac{2}{3}\, d\theta = \frac{4\pi}{3} \tag{4.7.6}$$

Ex. Find $\iint_S x dS$, where S is the triangle with vertices $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$.

Sol. The surface is a piece of a plane $ax+by+cz=d$ and putting in the 3 points we get $a=d$, $b=d$ and $c=d$, e.g. $a=b=c=d=1$. The surface is therefore given by $h(x,y,z) = x+y+z = 1$ and $(x,y)\in D = \{(x,y); x\ge 0, y\ge 0, x+y\le 1\}$. The normal to the surface is therefore $\mathbf{n} = \nabla h/|\nabla h| = (\mathbf{i}+\mathbf{j}+\mathbf{k})/\sqrt{3}$. We have

$$dS = \frac{dxdy}{|\cos\gamma|} = \frac{dxdy}{|\mathbf{n}\cdot\mathbf{k}|} = \sqrt{3}dxdy \tag{4.7.7}$$

If we rewrite $D = \{(x,y); 0\le x\le 1,\ 0\le y\le 1-x\}$ we get

$$\iint_S x\, dS = \iint_D x\sqrt{3}dxdy = \int_0^1\!\!\int_0^{1-x} x\sqrt{3}\, dy\, dx = \int_0^1 xy\sqrt{3}\Big|_{y=0}^{1-x} dx = \int_0^1 x(1-x)\sqrt{3}\, dx = \frac{\sqrt{3}}{6}$$

Surface Integrals of vector functions; The Flux. Let S be a closed surface and for each point (x, y, z) on the surface let $f(x, y, z)$ be the rate of flow of fluid out from the region enclosed by the surface per unit surface area and unit time, i.e. the flow out of a small area $\triangle S$ during a small time $\triangle t$ is approximately $f\,\triangle S \triangle t$. The total flow of fluid out from the region enclosed by the surface per unit time is

$$\iint_S f\,dS \tag{4.7.8}$$

according to (4.7.1). Let us now calculate the rate of flow of fluid f out per unit area and unit time, given the velocity vector field of the fluid $\mathbf{V}$ and the density μ. We define the **flow rate density** by

$$\mathbf{F} = \mu \mathbf{V}, \tag{4.7.9}$$

If $\triangle S$ is a small area of a piece of a plane with outward unit normal $\mathbf{n}$ then we claim that the flow rate out of $\triangle S$ per unit time is given by

$$\mathbf{F} \cdot \mathbf{n}\,\triangle S \tag{4.7.10}$$

In fact, in a small time $\triangle t$, the fluid particles that will reach $\triangle S$ are at most $\mathbf{V}\triangle t$ away, and all particles within reach form a sloped cylinder with $\triangle S$ as its base and height $\mathbf{V} \cdot \mathbf{n}\,\triangle t$. Since the volume is the area of the base times the height the amount of fluid in the cylinder is the density times the volume: $\mu\mathbf{V}\cdot\mathbf{n}\,\triangle t\triangle S$. If we divide by $\triangle t$ we get the rate per unit time (4.7.10) and if we divide this by $\triangle S$ we get the flow rate out of per unit surface area and unit time

$$f = \mathbf{F} \cdot \mathbf{n} \tag{4.7.11}$$

The flow rate of fluid out of the total surface S, or the **flux** of the vector field $\mathbf{F}$ out of the surface S, with outward unit normal $\mathbf{n}$, is given by

$$\iint_S \mathbf{F} \cdot \mathbf{n}\,dS \tag{4.7.12}$$