Lecture 17: 4.7 The Flux Cont.. Recall that we think the **surface integral** of a function f over the surface S as

$$\iint_S f\, dS = \lim_{\triangle S_{ij}\to 0} \sum_{i,j} f(x_{ij}, y_{ij}, z_{ij}) \triangle S_{ij} \tag{4.7.13}$$

where the sum is over a partition of S into smaller surface areas $\triangle S_{ij}$, (x_{ij}, y_{ij}, z_{ij}) is any point in $\triangle S_{ij}$ and we take the limit as the partition becomes finer.

Recall also the flow rate out a surface S, or the **flux** of the vector field $\mathbf{F}$ out of the surface S given by

$$\text{Flux}\,(\mathbf{F}) = \iint_S \mathbf{F}\cdot\mathbf{n}\; dS \tag{4.7.12}$$

where $\mathbf{n}$ is the outward unit normal.

The question of how to calculate the flux reduces how to calculate surface integrals. In a parametrization $\mathbf{R} = \mathbf{R}(u,v)$ we have $dS = |\mathbf{R}_u \times \mathbf{R}_v|\, dudv$ and $\mathbf{n} = \pm\mathbf{R}_u \times \mathbf{R}_v / |\mathbf{R}_u \times \mathbf{R}_v|$. Hence

$$\iint_S \mathbf{F}\cdot\mathbf{n}\; dS = \pm \iint \mathbf{F}\cdot(\mathbf{R}_u \times \mathbf{R}_v)\, dudv \tag{4.7.13}$$

Ex. Find the flux of $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} - 2z\,\mathbf{k}$ out of the surface S of the cube $C = \{(x,y,z);\, 0\le x\le 1,\, 0\le y\le 1,\, 0\le z\le 1\}$.

Sol. The Cube has six sides S_1 with $x=0$, S_2 with $x=1$, S_3 with $y=0$, S_4 with $y=1$, S_5 with $z=0$ and S_6 with $z=1$. On S_1, the outward normal is $-\mathbf{i}$ and $\mathbf{F}\cdot\mathbf{n} = (y\,\mathbf{j} - 2z\,\mathbf{k})\cdot(-\mathbf{i}) = 0$, on S_2, $\mathbf{F}\cdot\mathbf{n} = (\mathbf{i} + y\,\mathbf{j} - 2z\,\mathbf{k})\cdot\mathbf{i} = 1$, on $\mathbf{S}_3$, $\mathbf{F}\cdot\mathbf{n} = 0$, on S_4, $\mathbf{F}\cdot\mathbf{n} = 1$, on S_5, $\mathbf{F}\cdot\mathbf{n} = 0$, and on S_6, $\mathbf{F}\cdot\mathbf{n} = -2$. Since the area of each side is one it follows that

$$\iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \iint_{S_1} \mathbf{F}\cdot\mathbf{n}\, dS + \ldots + \iint_{S_6} \mathbf{F}\cdot\mathbf{n}\, dS = 0+1+0+1+0-2 = 0 \tag{4.7.14}$$

Ex. Find the flux of the vector field $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} - 2z\,\mathbf{k}$ out of the sphere $S = \{(x,y,z);\, x^2+y^2+z^2 = 1\}$.

Sol. The surface can be written $h(x,y,z) = x^2+y^2+z^2 = 1$. The outward unit normal to the unit sphere is $\mathbf{n} = \nabla h/|\nabla h| = (x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})/\sqrt{x^2+y^2+z^2} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, when $x^2+y^2+z^2 = 1$. Therefore

$$\iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \iint_S x^2 + y^2 - 2z^2\, dS$$

There are now several different ways to proceed. (1) Let us start with spherical coordinates, $\mathbf{R}(\phi,\theta) = \sin\phi\cos\theta\,\mathbf{i} + \sin\phi\sin\theta\,\mathbf{j} + \cos\phi\,\mathbf{k}$. Then we have seen that $dS = |\mathbf{R}_\phi \times \mathbf{R}_\theta|\, d\phi d\theta = \sin\phi\, d\phi d\theta$. Hence we get the integral

$$\begin{aligned}
&\int_0^{2\pi}\!\!\int_0^{\pi} \big(\sin^2\!\phi\cos^2\!\theta + \sin^2\!\phi\sin^2\!\theta - 2\cos^2\!\phi\big)\sin\phi\, d\phi\, d\theta \\
&\quad = \int_0^{2\pi}\!\!\int_0^{\pi} (\sin^2\phi - 2\cos^2\phi)\sin\phi\, d\phi d\theta = \int_0^{2\pi}\!\!\int_0^{\pi} (1 - 3\cos^2\phi)\sin\phi\, d\phi d\theta \\
&\quad = \int_0^{2\pi} -\cos\phi + \cos^3\phi\Big|_{\phi=0}^{\pi}\, d\theta = 0
\end{aligned}$$

(2) The sphere S can be written as the union of the northern hemisphere S_+ and southern hemisphere S_+ and each of these can be viewed as a graph over the x-y plane $S_+ = \{(x,y,z);\ z = \sqrt{1-x^2-y^2},\ (x,y) \in D\}$ and
$S_- = \{(x,y,z);\ z = -\sqrt{1-x^2-y^2},\ (x,y) \in D\}$, where $D = \{(x,y);\ x^2+y^2 \le 1\}$. Since the integrand and the sphere are symmetric under changing z to $-z$ we have

$$\iint_S x^2+y^2-2z^2\,dS = 2\iint_{S_+} x^2+y^2-2z^2 dS$$

We can now instead write $dS = \dfrac{dxdy}{|\cos\gamma|} = \dfrac{dxdy}{|\mathbf{n}\cdot\mathbf{k}|} = \dfrac{dxdy}{z} = \dfrac{dxdy}{\sqrt{1-x^2-y^2}}$ so

$$\iint_{S_+} x^2+y^2-2z^2 dS = \iint_D (3(x^2+y^2)-2)\frac{dxdy}{\sqrt{1-x^2-y^2}}$$

Introducing polar coordinates we get

$$\iint_D \frac{dxdy}{\sqrt{1-x^2-y^2}} = \int_0^{2\pi}\int_0^1 (3r^2-2)(1-r^2)^{-1/2} r\,dr\,d\theta$$

To find the anti derivative we write the integrand as
$3(r^2-1)(1-r^2)^{-1/2} + (1-r^2)^{-1/2} = -3(1-r^2)^{1/2} + (1-r^2)^{1/2}$ and hence get

$$\int_0^{2\pi} (1-r^2)^{3/2} - (1-r^2)^{1/2}\Big|_{r=0}^1 d\theta = 0$$

(3) Finally, one can also use symmetry to see that

$$\iint_S x^2 dS = \iint_S y^2\,dS = \iint_S z^2\,dS$$

and hence the integral we want to calculate vanishes.

4.8. We define the **Volume Integral** or **triple integral** of a function f over the volume V by

$$\iiint_V f\,dV = \lim_{\triangle V_{ijk}\to 0} \sum_{i,j,k} f(x_{ijk}, y_{ijk}, z_{ijk})\triangle V_{ijk}$$

where the sum is over a partition of V into smaller volumes $\triangle V_{ijk}$, $(x_{ijk}, y_{ijk}, z_{ijk})$ is any point in $\triangle V_{ijk}$ and we take the limit as the partition becomes finer.
If f is the density then the volume integral of f over V is the total mass in V.

From the definition above it is clear that

$$m\,\mathrm{Vol}(V) \le \iiint_V f\,dV \le M\,\mathrm{Vol}(V), \qquad \text{if} \quad m \le f \le M, \quad \text{in } V.$$

and in particular the volume integral of the constant function 1 over V is nothing but the volume of V.