Lecture 18: 4.8 Changing variables in Triple Integrals. Let $\mathbf{R}(u,v,w) = (x,y,z)$ be an invertible mapping $x = x(u,v,w)$, $y = y(u,v,w)$ and $z = z(u,v,w)$ of a region $\tilde{D}$ in (u,v,w)-space onto a region D in (x,y,z) space. Then

$$\iiint_D f(x,y,z)\,dxdydz = \iiint_{\tilde{D}} f\big(x(u,v,w), y(u,v,w), z(u,v,w)\big)\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dudvdw,$$

where the Jacobian determinant is given by

$$\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}$$

This result can symbolically be written

$$dxdydz = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dudvdw$$

In particular spherical coordinates are given by
$\mathbf{R}(r,\phi,\theta) = r\sin\phi\cos\theta\,\mathbf{i} + r\sin\phi\sin\theta\,\mathbf{j} + r\cos\phi\,\mathbf{k}$, $r \ge 0$, $0 \le \phi \le \pi$ and $0 \le \theta < 2\pi$.
We have

$$dV = dxdydz = r^2\,\sin\phi\,dr\,d\phi d\theta$$

Recall from a previous section that the surface are element on the spheres of radius are is

$$dS = r^2\,\sin\phi\,d\phi d\theta$$

Hence

$$dV = drdS,$$

i.e. the volume element is the area element dS times the thickness dr. One can actually see the relation between the volume element and the surface area element directly form the definition. In fact

$$dS = |\mathbf{R}_\phi \times \mathbf{R}_\theta| d\phi d\theta$$

and, since the scalar tripple product is the same as the determinant

$$dV = \big|\big(\mathbf{R}_\phi \times \mathbf{R}_\theta\big)\cdot \mathbf{R}_r\big|\,d\phi d\theta\,dr$$

because $\mathbf{R}_\phi \times \mathbf{R}_\theta = N$ is normal to the spheres r constant and a calculation shows that $\mathbf{R}_r$ is unit normal to the surface so

$$\big|\big(\mathbf{R}_\phi \times \mathbf{R}_\theta\big)\cdot \mathbf{R}_r\big| = |\mathbf{R}_\phi \times \mathbf{R}_\theta|$$

Ex. Find the area of the unit ball $B = \{(x,y,z);\, x^2+y^2+z^2 \le 1\}$.

$$\iiint_B 1\,dxdydz = \int_0^{2\pi}\int_0^{\pi}\int_0^1 r^2\,\sin\phi\,dr\,d\phi d\theta = \int_0^{2\pi}\int_0^{\pi} \frac{r^3}{3}\Big|_0^1 \sin\phi\,d\phi d\theta$$

$$\int_0^{2\pi}\int_0^{\pi} \frac{\sin\phi}{3}\,d\phi d\theta = \int_0^{2\pi} -\frac{\cos\phi}{3}\Big|_0^{\pi}\,d\theta \int_0^{2\pi} \frac{2}{3}\,d\theta = \frac{4\pi}{3}$$

4.9. The divergence theorem states that if V is a volume bounded by a surface S with outward unit normal $\mathbf{n}$ and $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ is a vector fields then

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_V \operatorname{div}\mathbf{F}\,dV, \qquad \text{where} \quad \operatorname{div}\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

Ex. Find flux of $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} - 2z\,\mathbf{k}$ out of the unit sphere.

Sol. By the divergence theorem the flux is equal to the integral of the divergence over the unit ball. Since $\operatorname{div}\mathbf{F} = 0$ it follows that the volume integral vanishes and by the divergence theorem the flux therefore vanishes.

Let us prove this theorem for a rectangular box with sides lengths ε: $B = \{(x,y,z);\ 0 \le x \le \varepsilon,\ 0 \le y \le \varepsilon,\ 0 \le z \le \varepsilon\}$. The surface S is the union of the six sides $S_1 = \{(x,y,z); x = 0,\ 0 \le y \le \varepsilon\, 0 \le z \le \varepsilon\}$, $S_2 = \{(x,y,z); x = \varepsilon,\ 0 \le y \le \varepsilon\, 0 \le z \le \varepsilon\}$ and etc. for S_3, S_4, S_5 and S_6. The flux is therefore

$$\iint_{S_\varepsilon} \mathbf{F}\cdot\mathbf{n}\,dS = \iint_{S_1} \mathbf{F}\cdot\mathbf{n}\,dS + \iint_{S_2} \mathbf{F}\cdot\mathbf{n}\,dS + \ldots + \iint_{S_6} \mathbf{F}\cdot\mathbf{n}\,dS$$

At S_1 the outward unit normal is $\mathbf{n} = -\mathbf{i}$ so $\mathbf{F}\cdot\mathbf{n} = -F_1$ and at S_2 he outward unit normal is $\mathbf{n} = -\mathbf{i}$ so $\mathbf{F}\cdot\mathbf{n} = F_1$. Therefore

$$\begin{aligned}\iint_{S_2} \mathbf{F}\cdot\mathbf{n}\,dS + \iint_{S_1} \mathbf{F}\cdot\mathbf{n}\,dS &= \int_0^\varepsilon\int_0^\varepsilon F_1(\varepsilon,y,z)\,dydz - \int_0^\varepsilon\int_0^\varepsilon F_1(0,y,z)\,dydz \\ &= \int_0^\varepsilon\int_0^\varepsilon F_1(\varepsilon,y,z) - F_1(0,y,z)\,dydz \\ &= \int_0^\varepsilon\int_0^\varepsilon\int_0^\varepsilon \frac{\partial F_1}{\partial x}(x,y,z)\,dx\;dydz = \iiint_{B_\varepsilon} \frac{\partial F_1}{\partial x} dV\end{aligned}$$

Similarly adding the surface integral over the surfaces with $y = 0$ and $y = \varepsilon$ gives $\iiint_{B_\varepsilon} \partial F_2/\partial y dV$ and adding the surface integral over the surfaces with $z = 0$ and $z = \varepsilon$ gives $\iiint_{B_\varepsilon} \partial F_3/\partial z dV$. Therefore

$$\iint_{S_\varepsilon} \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_{B_\varepsilon} \operatorname{div}\mathbf{F}\,dV$$

By the mean value theorem for integrals the right hand side is equal to the volume of the box B_ε times $\operatorname{div}\mathbf{F}$ at some point in the box so we get the interpretation of the divergence that we announced in section 3.3:

$$\text{Flux of } \mathbf{F} \text{ out through } S_\varepsilon = \operatorname{Vol}(B_\varepsilon)\operatorname{div}\mathbf{F}$$

where $\operatorname{div}\mathbf{F}$ is evaluated at some point in B_ε. Hence

$$\operatorname{div}\mathbf{F} = \lim_{\varepsilon\to 0} \frac{\text{Flux of } \mathbf{F} \text{ out through } S_\varepsilon}{\operatorname{Vol}(B_\varepsilon)}$$