Lecture 21: Section 5.4: Gren's theorem. Suppose that D is a domain in the plane with boundary curve C going in positive direction, i.e. walking in the direction of C the domain D should be on your left. **Green's theorem** says that

$$\int_C P dx + Q dy = \iint_D \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big) dx dy \tag{5.4.1}$$

Ex Let $\mathbf{F} = xy\mathbf{i} + y^2\mathbf{j}$ and let D be the region between the graph of $y = x^2$ and $y = x$ when $0 \le x \le 1$. Let C be the positively oriented boundary of D. Calculate both sides of Green's theorem and check that it holds.

Sol. The boundary consists of two parts C_1; $x = t$, $y = t^2$, $0 \le t \le 1$ and C_2; $x = 1-t$, $y = 1-t$, $0 \le t \le 1$. Note that C_2 is oriented so it starts at $(x, y) = (1, 1)$ in order that the total curve should be positively oriented. Let $P = xy$ and $Q = y^2$.

$$\int_C P\,dx + Q\,dy == \int_{C_1} \Big(xy\frac{dx}{dt} + y^2\frac{dy}{dt}\Big)\,dt + \int_{C_2} \Big(xy\frac{dx}{dt} + y^2\frac{dy}{dt}\Big)\,dt$$

$$= \int_0^1 t^3 + 2t^5\,dt + \int_0^1 -(1-t)^2 - (1-t)^2\,dt = \frac{t^4}{4} + \frac{t^6}{3}\Big|_0^1 + \frac{2(1-t)^3}{3}\Big|_0^1 = \frac{1}{4} + \frac{1}{3} - \frac{2}{3} = -\frac{1}{12}$$

On the other hand

$$\iint_D \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\,dx dy = \iint_D -x\,dx dy = \int_0^1 \int_{x^2}^{x} -x\,dy\,dx = \int_0^1 -x(x - x^2)\,dx$$

$$= -\frac{x^3}{3} + \frac{x^4}{4}\Big|_0^1 = -\frac{1}{3} + \frac{1}{4} = -\frac{1}{12}$$

Ex. Find $\int_C -y\,dx + x\,dy$, where C is the unit circle centered at $(0, 0)$.

Sol. Let D be the unit disc centered at $(0, 0)$. By Green's theorem

$$\int_C -y\,dx + x\,dy = \iint_D \Big(\frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y}\Big)\,dx dy = \iint_D 2\,dx dy = 2\text{Area}\,(D) = 2\pi$$

From the proof of the above example we conclude that for any region D we have

$$\text{Area}\,(D) = \frac{1}{2}\int_C -y\,dx + x\,dy \text{Area}\,(D) = \int_C -y\,dx = \int_C x\,dy \tag{5.4.2}$$

which gives another way to calculate the area. Say that D is a region of type 1, i.e. lying above the graph: C_1: $y = f_1(x)$, $a \le x \le b$ and below the graph C_2: $y = f_2(x)$, $b \ge x \ge a$. Then, the area below C_2 and above the x-axis is $\int_a^b f_2(x)\,dx = -\int_b^a f_2(x)\,dx = \int_{C_2} -y\,dx$ and the area below C_1 and above the x-axis is $\int_a^b f_1(x)\,dx = \int_{C_1} y\,dx$, since y is the height above the x-axis. Therefore the area of D is $\int_a^b f_2(x)\,dx - \int_a^b f_1(x)\,dx = \int_{C_1} -y\,dx + \int_{C_2} -y\,dx$.

Ex. Find the area of the interior of an ellipse: $D = \{(x, y);\ (x/a)^2 + (y/b)^2 \le 1\}$

Sol. Parametrizing the ellipse $x = a\cos t$, $y = b\sin t$, $0 \le t \le 2\pi$ and using (5.4.2)

$$\text{Area}\,(D) = \frac{1}{2}\int_C -y\,dx + x\,dy = \frac{1}{2}\int_0^{2\pi} -y\frac{dx}{dt} + x\frac{dy}{dt}\,dt$$

$$= \frac{1}{2}\int_0^{2\pi} -b\sin t(-a\sin t) + (a\cos t)b\sin t\,dt = \frac{1}{2}\int_0^{2\pi} ab(\sin^2 t + \cos^2 t)\,dt = \pi ab$$

Let us see how Green's theorem implies the divergence theorem in two dimensions:

$$\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \iint_D \operatorname{div}\mathbf{F}\,dxdy, \qquad \mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j}$$

If the curve C is parametrized by arclength $x = x(s)$ and $y = y(s)$, $a \le s \le b$, then since the normal $\mathbf{n}$ is perpendicular to the unit tangent vector (x', y') it follows that unit normal is given by $(n_1, n_2) = (y', -x')$. Hence $dx = x'ds$ and $dy = x'ds$ and with $P = -F_2$ and $Q = F_1$ we get

$$\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_C Pdx + Qdy$$

and

$$\iint_D \operatorname{div}\mathbf{F}\,dxdy = \iint_D \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)dxdy$$

Let us also see how Stokes's theorem in a special case follows from Green's theorem. Suppose that D is a reion in the x-y plane with positively oriented boundary and suppose that $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j}$ where $F_1 = P$ and $F_2 = Q$, where P and Q are independent of z. Then

$$\mathbf{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial x} \\ P & Q & 0 \end{vmatrix} = \mathbf{k}\Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big), \qquad \mathbf{F}\cdot\mathbf{n}$$

It follows that

$$\int_C \mathbf{F}\cdot\mathbf{R} = \int_C Pdx + Qdy, \qquad \iint_D \mathbf{curl}\,\mathbf{F}\cdot\mathbf{n}\,dxdy = \iint_D \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)dxdy$$

And hence a special case of Stokes theorem follows from Greens theorem

$$\int_C \mathbf{F}\cdot\mathbf{dR} = \iint_D \mathbf{curl}\,\mathbf{F}\cdot\mathbf{n}\,dS$$

Let us now finally discuss how to prove Green's theorem for a region of type 3, which means it is a region of type 1: $D = \{(x,y); f_1(x) \le y \le f_2(x)\}$, $a \le x \le b\}$ and a region of type 2: $D = \{(x,y); g_1(y) \le x \le g_2(y)\}$, $c \le y \le d\}$. Then

$$\iint_D \frac{\partial P}{\partial y}\,dxdy = \int_a^b \int_{f_1(x)}^{f_2(x)} \frac{\partial P}{\partial y}(x,y)\,dy\;dx = \int_a^b P(x, f_2(x))\,dx - \int_a^b P(x, f_1(x))\,dx$$
$$= -\int_b^a P(x, f_2(x))\,dx - \int_a^b P(x, f_1(x))\,dx = -\int_C Pdx.$$

Similarly

$$\iint_D \frac{\partial Q}{\partial x}\,dxdy = \int_c^d \int_{g_1(y)}^{f_2(y)} \frac{\partial Q}{\partial x}(x,y)\,dx\;dy = \ldots = \int_C Qdy.$$