Lecture 22: Section 5.5. Let S be a surface with unit normal $\mathbf{n}$ and boundary C positively oriented, i.e. if you walk in the direction of the curve on the side of the normal then you should have the surface on your left. Stokes' theorem says

$$\int_C \mathbf{F}\cdot d\mathbf{R} = \iint_S \mathbf{curl}\,\mathbf{F}\cdot\mathbf{n}\,dS$$

Last time we showed that in the special case that S is a domain in the x-y plane Stokes' theorem follows from Green's theorem. One can show that both sides in Stokes' theorem are invariant under rotations and one can rotate any plane to the x-y plane. Stokes theorem for any domain of any plane therefore follows. An intuitive proof in general follows by approximating the surface with a net of small triangles.

We will first prove Stokes' theorem for a graph $z = f(x,y)$, $(x,y)\in D$. Since the surface can be written $h(x,y,z) = z - f(x,y)$ a normal is given by $\mathbf{N} = \nabla h = -f_x\mathbf{i} - f_y\mathbf{j} + \mathbf{k}$ and the unit normal is given by $\mathbf{n} = \mathbf{N}/|\mathbf{N}|$. The surface measure is $dS = dxdy/|\mathbf{k}\cdot\mathbf{n}|$, where $\mathbf{k}\cdot\mathbf{n} = \mathbf{k}\cdot\mathbf{N}/|\mathbf{N}| = 1/|\mathbf{N}|$, so $dS = |\mathbf{N}|\,dxdy$ and hence

$$\iint_S \mathbf{G}\cdot\mathbf{n}\,dS = \iint_D -G_1 f_x - G_2 f_y + G_3\,dxdy, \qquad \text{if}\quad \mathbf{G} = G_1\mathbf{i} + G_2\mathbf{j} + G_3\mathbf{k}$$

If we apply to $\mathbf{F}$ this to $\mathbf{G} = \mathbf{curl}\,\mathbf{F}$ we get

$$\iint_S \mathbf{curl}\,\mathbf{F}\cdot\mathbf{n}\,dS = \iint_D -\Big(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\Big)f_x - \Big(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\Big)f_y + \Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big)\,dxdy,$$

If we parametrize the boundary $x = x(t)$, $y = y(t)$ and $z = f(x,y)$ we have by the chain rule

$$\frac{dz}{dt} = f_x\frac{dx}{dt} + f_y\frac{dy}{dt}$$

and

$$\int_C \mathbf{F}\cdot d\mathbf{R} = \int_a^b \Big(F_1\frac{dx}{dt} + F_2\frac{dy}{dt} + F_3\frac{dz}{dt}\Big)\,dt = \int_a^b \Big((F_1 + f_xF_3)\frac{dx}{dt} + (F_2 + f_yF_3)\frac{dy}{dt}\Big)\,dt$$

This can now be considered as a line integral in the plane:

$$\int_C \mathbf{F}\cdot d\mathbf{R} = \int_{\partial D} P dx + Q\,dy, \qquad \text{where}$$

$$P(x,y) = F_1(x,y,f(x,y)) + f_x(x,y)F_3(x,y,f(x,y)),$$
$$Q(x,y) = F_2(x,y,f(x,y)) + f_y(x,y)F_3(x,y,f(x,y))$$

We can therefore apply Greens formula in the plane.

$$\frac{\partial P}{\partial y} = \frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}f_y + f_x\frac{\partial F_3}{\partial y} + f_xf_y\frac{\partial F_3}{\partial z} + f_{xy}F_3$$
$$\frac{\partial Q}{\partial x} = \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial z}f_x + f_y\frac{\partial F_3}{\partial x} + f_xf_y\frac{\partial F_3}{\partial z} + f_{xy}F_3$$

so by Green's theorem

$$\int_C \mathbf{F}\cdot\mathbf{dR} = \int_{\partial D} P dx + Q\,dy = \iint_D \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\,dxdy$$
$$= \iint_D -\Big(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\Big) f_x - \Big(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\Big) f_y + \Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big)\,dxdy,$$

Ex. Show that $\int_C ye^z dx + xe^z dy + xye^z dz = 0$ for a closed curve C.
Sol. $\mathbf{F} = \nabla(xye^z)$ so $\mathbf{curl}\,\mathbf{F} = 0$ and by Stokes's theorem the integral vanishes.
Ex. Find $\int_{C_a} \mathbf{F}\cdot\mathbf{dR}$, where $\mathbf{F} = (-y\mathbf{i}+x\mathbf{j})/(x^2+y^2)$ and C_a is the circle $x^2+y^2 = a^2$ in the x-y plane going counterclockwise.
Sol. $\mathbf{curl}\,\mathbf{F} = ... = 0$. Hence one would have thought that by Stokes theorem the line integral would vanish. However, if we parametrize $x = a\cos t$ and $y = a\sin t, 0 \le t < 2\pi$, we get

$$\int_{C_a} \mathbf{F}\cdot\mathbf{dR} = \int_0^{2\pi} \Big(\frac{-y}{x^2+y^2}\frac{dx}{dt} + \frac{x}{x^2+y^2}\frac{dy}{dt}\Big)\,dt$$
$$= \int_0^{2\pi} \Big(\frac{-a\sin t(-a\sin t)}{a^2} + \frac{a\cos t(a\cos t)}{a^2}\Big)\,dt = \int_0^{2\pi} \sin^2 t + \cos^2 t\,dt = 2\pi$$

The reason Stokes' theorem failed to hold in this case was that the vector field $\mathbf{F}$ is singular when $(x,y) = (0,0)$, i.e. along the z-axis.