

Lecture 2: Section 1.12.: Let $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$ and $\mathbf{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$ be two vectors making an angle θ . The **scalar or dot product** is the scalar

$$(1.12.0) \quad \mathbf{A} \cdot \mathbf{B} = A_1B_1 + A_2B_2 + A_3B_3 = |\mathbf{A}| |\mathbf{B}| \cos \theta$$

The **vector or cross product** is the vector

$$(1.12.1) \quad \mathbf{A} \times \mathbf{B} = (A_2B_3 - A_3B_2)\mathbf{i} + (A_3B_1 - A_1B_3)\mathbf{j} + (A_1B_2 - A_2B_1)\mathbf{k} = |\mathbf{A}| |\mathbf{B}| \sin \theta \mathbf{N}.$$

Here \mathbf{N} is a unit vector, i.e. $|\mathbf{N}| = 1$, that is perpendicular to both \mathbf{A} and \mathbf{B} and pointing in the direction so that \mathbf{A} , \mathbf{B} and \mathbf{N} form a positively oriented system. First we note that the scalar product is a scalar whereas the vector product is a vector. Secondly, just like for the scalar product there are two possible definitions of the vector product in (1.12.1) above, and it is a theorem that they are equal. One could start with either of the last two quantities in (1.12.1) as the definition of the cross product and prove that it is equal to the other quantity.

The last expressions in (1.12.0) and (1.12.1) has a clear geometry interpretation. $|\mathbf{B}| \cos \theta$ is the component of \mathbf{B} along or in the direction of \mathbf{A} , also called the scalar projection of \mathbf{B} onto \mathbf{A} . $|\mathbf{B}| \sin \theta$ is the component of \mathbf{B} perpendicular to \mathbf{A} . Therefore it follows that $|\mathbf{A} \times \mathbf{B}| = |\mathbf{A}| |\mathbf{B}| \sin \theta$ is the area of the parallelogram with \mathbf{A} and \mathbf{B} as two of its sides. Also it follows from the last expressions that $\mathbf{A} \cdot \mathbf{B} = 0$ if and only if \mathbf{A} and \mathbf{B} are perpendicular. Furthermore it follows that the cross product $\mathbf{A} \times \mathbf{B}$ is perpendicular to both \mathbf{A} and \mathbf{B} . Also $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$.

Say that we know the scalar product but don't know the cross product and we need to find a vector $\mathbf{N} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ that is perpendicular to both \mathbf{A} and \mathbf{B} . Then the vector must satisfy:

$$(1.12.2) \quad \mathbf{A} \cdot \mathbf{N} = 0, \quad \mathbf{B} \cdot \mathbf{N} = 0$$

or if we write it out

$$(1.12.3) \quad A_1a + A_2b + A_3c = 0, \quad B_1a + B_2b + B_3c = 0$$

This is a system of two equations and one three unknowns so the solution (a, b, c) is a line. Multiplying the first equation by B_1 and the second by A_1 we get

$$(1.12.4) \quad B_1A_1a + B_1A_2b + B_1A_3c = 0, \quad A_1B_1a + A_1B_2b + A_1B_3c = 0$$

Subtracting the second from the first we get

$$(1.12.5) \quad (A_2B_1 - A_1B_2)b + (A_3B_1 - A_1B_3)c = 0$$

Hence

$$(1.12.6) \quad b = -\frac{A_3B_1 - A_1B_3}{A_2B_1 - A_1B_2}c$$

If we pick $c = -A_2B_1 + A_1B_2$ then $b = A_3B_1 - A_1B_3$ and

$$a = -(A_2b + A_3c)/A_1 = -(A_2(A_3B_1 - A_1B_3) + A_3(-A_2B_1 + A_1B_2))/A_1 = A_2B_3 - A_3B_2$$

We hence obtained $\mathbf{N} = (A_2B_3 - A_3B_2)\mathbf{i} + (A_3B_1 - A_1B_3)\mathbf{j} + (A_1B_2 - A_2B_1)\mathbf{k}$ which is exactly (1.12.1).

The first expression in (1.12.1) however appears strange and hard to remember, but on the other hand this is the one that is easiest to use when one actually calculates the cross product.

In order to remember the definition (1.12.1) we introduce so called **determinants**. A **determinant of order 2** is defined by

$$(1.12.7) \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Geometrically, this is plus or minus the **area** of the parallelogram in the plane with the vectors $\langle a, b \rangle$ and $\langle c, d \rangle$ as two edges. (This will be shown shortly.)

A **determinant of order 3** is defined by

$$(1.12.8) \quad \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Geometrically, this is plus or minus the **volume** of the parallelepiped with the vectors $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$, $\mathbf{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$ and $\mathbf{C} = C_1\mathbf{i} + C_2\mathbf{j} + C_3\mathbf{k}$ as three edges.

The cross product (1.12.1) is

$$(1.12.9) \quad \mathbf{A} \times \mathbf{B} = \begin{vmatrix} A_2 & A_3 \\ B_2 & B_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} A_1 & A_3 \\ B_1 & B_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} A_1 & A_2 \\ B_1 & B_2 \end{vmatrix} \mathbf{k}$$

Because of the similarity with (1.12.8), to remember this we symbolically write

$$(1.12.10) \quad \mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

We also note that using (1.12.8) and (1.12.9) it follows that the so called **scalar triple product** is equal to the determinant

$$(1.12.11) \quad \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

Ex. If $\mathbf{A} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{B} = -\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$ then

$$(1.12.12) \quad \begin{aligned} \mathbf{a} \times \mathbf{B} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ -1 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 2 & 4 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 3 \\ -1 & 4 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix} \mathbf{k} \\ &= (2 \cdot 4 - 3 \cdot 2)\mathbf{i} - (1 \cdot 4 - 3 \cdot (-1))\mathbf{j} + (1 \cdot 2 - 2 \cdot (-1))\mathbf{k} = 2\mathbf{i} - 7\mathbf{j} + 4\mathbf{k} \end{aligned}$$

Volume of a parallelepiped Consider the parallelepiped determined by \mathbf{A} , \mathbf{B} and \mathbf{C} . Recall that $A = |\mathbf{B} \times \mathbf{C}|$ is the area of the parallelogram with \mathbf{B} and \mathbf{C} as edges. If θ is the angle between \mathbf{a} and $\mathbf{b} \times \mathbf{c}$ then $h = |\mathbf{A}| |\cos \theta|$ is the height of the parallelepiped. Therefore the volume is

$$(1.12.13) \quad V = Ah = |\mathbf{B} \times \mathbf{C}| |\mathbf{A}| |\cos \theta| = |\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})|.$$

Ex. Find the volume of a parallelepiped determined by $\mathbf{A} = \mathbf{i} - 2\mathbf{j}$, $\mathbf{B} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{C} = -\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$.

Sol. By previous example $V = |\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})| = |(\mathbf{i} - 2\mathbf{j}) \cdot (2\mathbf{i} - 7\mathbf{j} + 4\mathbf{k})| = |2 + 14| = 16$.