Lecture 6: Section 3.2. A **Vector field** $\mathbf{F}$ is a rule that to each point (x, y, z) in space assigns a vector $\mathbf{F}(x, y, z) = F_1(x, y, z)\mathbf{i} + F_2(x, y, z)\mathbf{j} + F_3(x, y, z)\mathbf{k}$. Mathematically, a vector field is a map $\mathbf{F} : \mathbf{R}^3 \to \mathbf{R}^3$ and the components are just scalar valued functions. Graphically, a vector field is illustrated by, from a few points in space $\mathbf{R}$ drawing the arrow representing the vector $\mathbf{F}(\mathbf{R})$. There are many physical examples of vector fields, e.g. the gravitational field at each point in space by Newton's law tells the mass in which direction to accelerate and how much. Similarly, one can describe a charge moving in an electric field. Another example is a sand storm, where the vector field at a point in space $\mathbf{F}(\mathbf{R})$ gives the velocity of the sand particle at the point $\mathbf{R}$. Similarly, the velocity vector field of a fluid gives the velocity of the fluid (particle) at each point in space.

A **Flow line** for a vector field $\mathbf{F}$ is a curve $\mathbf{R}(t)$ such that at each point along the curve, the vector field $\mathbf{F}(\mathbf{R})$ is tangent to the curve and in the same direction as the curve $\mathbf{R}'(t)$. If the vector field is the velocity vector field of a fluid then the flow lines are just the curves along which the fluid particles travel. From the graphic illustration of a vector field one can approximately draw the flow lines, by going in the direction of the vector field. If β is a positive scalar field then the flow lines for $\beta\mathbf{F}$ are the same as the flow lines for $\mathbf{F}$ since the flow lines are determined by the direction of the vector field. If $\mathbf{R}$ is a flow line then for some positive function β:

$$\frac{d\mathbf{R}}{dt} = \beta\mathbf{F}, \qquad \beta \geq 0 \tag{3.2.1}$$

The flow lines can not intersect since the vector field gives the direction. Furthermore, through each point there is a unique flow line. In fact through each initial point there is a unique solutions to the differential equation (3.2.1). Moreover, one can make $\beta = 1$ by changing the parameter along the curve and this only changes the speed at which we travel but the curve along which we travel is the same. In order to solve (3.2.1): $dx/dt = \beta F_1$, $dy/dt = \beta F_2$, $dz/dt = \beta F_3$ one can eliminate β:

$$\frac{dx}{F_1} = \frac{dy}{F_2} = \frac{dz}{F_3} \tag{3.2.2}$$

Ex. 3.2.1 Find the flow lines for the vector field $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$.
Sol. $dx/x = dy/y = dz/0$. The last equation should be interpreted as $dz = 0$ so $z = C_1$. The second equation $dx/x = dy/y$ gives $\ln x = \ln y - C$ so if we exponentiate $y = xe^C = C_2 x$. The flow lines are therefore half lines going out from the z-axis
Ex. 3.2.2 Find the flow lines for the vector field $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$.
Sol.$-dx/y = dy/x = dz/0$. Hence $dz = 0$ so $z = c_1$. Also $-dx/y = dy/x$ so $-xdx = ydy$ and $-x^2/2 = y^2/2 - c$, i.e. $x^2 + y^2 = c_2$. The flow lines are circles around the z-axis. One can check that the parametrization for the circle $x = r\cos t$, $y = r\sin t$, $z = c$ satisfies the equations of the flow line $dx/dt = -y$, $dy/dt = x$, $dz/dt = 0$.

Section 3.3. We are now about to take derivatives of a vector field. Since the vector field have three components and each depends on three variables we are faced with trying to interpret the meaning of nine different derivatives. However, it turns out that certain combinations of these derivatives have a clear geometric and physical meaning. One is the **divergence** of a vector field which is a scalar field and the other is the **curl** of a vector field which is a vector field. The divergence tells us to what extent the field is spreading the particles out, "diverging" (Ex. 3.2.1). The curl tells us how the vector field "swirls" around (Ex. 3.2.2)

The divergence is defined by

$$\operatorname{div}\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial y}, \qquad \text{if} \qquad \mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k} \tag{3.3.1}$$

Ex. 3.3.1 If $\mathbf{F} = x\mathbf{i}+y\mathbf{j}$ then $\operatorname{div}\mathbf{F} = 2$. **Ex. 3.3.2** If $\mathbf{F} = -y\mathbf{i}+x\mathbf{j}$ then $\operatorname{div}\mathbf{F} = 0$.

From these two examples it appears that the description of divergence as how much the flow lines go apart is correct. However, there is more to it:

Ex. 3.3.3 Let $\mathbf{F}=(x\mathbf{i} + y\mathbf{j})/(x^2 + y^2)$. The $\operatorname{div}\mathbf{F}=0$.

The flow lines in Ex. 3.3.3 are the same as in Ex. 3.3.1 since only the magnitude of the vector field changed, i.e., by Ex. 3.2.1, lines that go out from the origin. The magnitude of the vector field in Ex. 3.3.3 however decreases as we go out. The flow lines do not depend on the magnitude of the vector field but the divergence does.

If the vector field is the velocity vector field of a fluid then the divergence is the rate of expansion of the fluid per unit volume. If $\mathbf{V}$ is the velocity vector field of the fluid and $\mathbf{R}$ is the position of a fluid particle then

$$\frac{d\mathbf{R}}{dt} = \mathbf{V}(\mathbf{R}) \tag{3.3.2}$$

If we follow the positions of all fluid particles within a small domain we get a domain $\mathcal{D}_t$ depending on time t. The rate of change of the volume $\operatorname{Vol}(\mathcal{D}_t)$ of this domain is

$$\frac{d\operatorname{Vol}(\mathcal{D}_t)}{dt} = \operatorname{Vol}(\mathcal{D}_t)\ \operatorname{div}\mathbf{V}, \tag{3.3.3}$$

where $\operatorname{div}\mathbf{V}$ is evaluated at some point in $\mathcal{D}_t$. This follows from the transport theorem, sections 4.10 and 5.6. (3.3.3) says that the divergence is the rate of expansion of the fluid volume per unit volume. An incompressible liquid is divergence free $\operatorname{div}\mathbf{V}=0$ whereas a gas is compressible and the divergence is nonvanishing.

If the fluid expands then the average density must decrease and fluid must flow out of any fixed region. If we instead consider the amount of fluid $\mathcal{M}_t$ in a small fixed domain $\mathcal{D}$ and define the **flow rate density** by

$$\mathbf{F} = \mu\mathbf{V}, \tag{3.3.4}$$

where μ is the density, then the rate of change of the amount of fluid in $\mathcal{D}$ is

$$\frac{d\mathcal{M}_t}{dt} = -\operatorname{Vol}(\mathcal{D})\ \operatorname{div}\mathbf{F} \tag{3.3.5}$$

where $\operatorname{div}\mathbf{F}$ is evaluated at some point in $\mathcal{D}$. This follows from the divergence theorem, sections 4.9 and 5.1. The rate of change of the amount of fluid in the domain is equal to the amount of fluid that goes out through the surface $\mathcal{S}$ of the domain per unit time. The amount of fluid that goes out through the surface $\mathcal{S}$ per unit time is is called the **flux** or **rate of flow** of the vector field $\mathbf{F}$ through $\mathcal{S}$. We will calculate the flux later using surface integrals and here we only give a brief description how it can be calculated. If $\triangle S$ is a small area of a piece of a plane with outward unit normal $\mathbf{n}$ then we claim that the flow rate out of $\triangle S$ is given by

$$\mathbf{F}\cdot\mathbf{n}\,\triangle S \tag{3.3.6}$$

In fact, in a small time $\triangle t$, the fluid particles that will reach $\triangle S$ are at most $\mathbf{V}\triangle t$ away, and all particles within reach form a sloped cylinder with $\triangle S$ as its base and height $\mathbf{V}\cdot\mathbf{n}\,\triangle t$. Since the volume is the area of the base times the height the amount of fluid in the cylinder is the density times the volume: $\mu\mathbf{V}\cdot\mathbf{n}\,\triangle t\triangle S$. (3.3.6) is the rate per unit time is. In the book it is proved that for a small rectangular box:

$$\text{Flux of } \mathbf{F} \text{ out through } \mathcal{S} = \operatorname{Vol}(\mathcal{D})\ \operatorname{div}\mathbf{F} \tag{3.3.7}$$