Lecture 9: Section 4.3: Conservative fields-the potential. We say that a vector field is **conservative** if there is a scalar field ϕ, called **potential**, such that

$$\mathbf{F} = \mathbf{grad}\ \phi \tag{4.3.1}$$

Theorem 4.3.1. *A vector field* $\mathbf{F}$ *in a domain* D *is conservative if and only if the line integral of* $\mathbf{F}$ *along every regular curve in* D *depends only on the endpoints of the curve. In that case, the integral is the difference in potential of the endpoints:*

$$\int_C \mathbf{F}\cdot d\mathbf{R} = \phi(Q) - \phi(P) \tag{4.3.2}$$

where P *and* Q *are the initial and endpoints of* C.

In physical terms the theorem says that the work is independent of the way. That the line integral is independent of the way is equivalent to that the line integral along any closed path or loop vanishes.

Ex. Let $\mathbf{F} = y\mathbf{i} + x\mathbf{j}$. Find a potential function and evaluate the line integrals in Ex. 4.1.2 using the potential function and Theorem 4.3.1.

Sol. We want to find ϕ so that $\partial\phi/\partial x = y$, $\partial\phi/\partial y = x$ and $\partial\phi/\partial z = 0$. Integration of $\partial\phi/\partial x = y$ gives $\phi = xy + g(y,z)$, where g is any function of y and z. With this ϕ it follows that $\partial\phi/\partial y = x$ and $\partial\phi/\partial z = 0$ if $g(y,z) = C$ is a constant. Hence $\phi = xy + C$, for any constant C, satisfies $\mathbf{grad}\ \phi = \mathbf{F}$. By Th. 4.3.1:

$$\int_{C_1} \mathbf{F}\cdot d\mathbf{R} = \int_{C_1} \mathbf{F}\cdot d\mathbf{R} = \phi(1/\sqrt{2}, 1/\sqrt{2}, 0) - \phi(1,0,0) = \frac{1}{2} \tag{4.3.3}$$

Note that $\mathbf{F} = \mathbf{grad}\ \phi$ is perpendicular to the level surfaces of ϕ and hence the level surfaces of the potential function are perpendicular to the flow lines of $\mathbf{F}$.

Proof of Th If $\mathbf{F} = \mathbf{grad}\ \phi$ then we have for any curve C from P to Q:

(4.3.4)

$$\int_C \mathbf{F}\cdot d\mathbf{R} = \int_a^b \frac{\partial\phi}{\partial x}\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\frac{dy}{dt} + \frac{\partial\phi}{\partial z}\frac{dz}{dt}\, dt = \int_a^b \frac{d}{dt}\phi(x(t),y(t),z(t))\, dt = \phi(Q) - \phi(P),$$

by the chain rule. Assume that it is independent of the way, then we can define

$$\phi(x,y,z) = \int_{(x_0,y_0,z_0)}^{(x,y,z)} \mathbf{F}\cdot d\mathbf{R} \tag{4.3.5}$$

where the integral is along any curve form a fixed point (x_0, y_0, z_0) to (x,y,z). This is well-defined since it is independent of the curve and since the domain is connected so there is such a curve. It also follows that for any (x',y',z') and (x,y,z) we have

$$\phi(x',y',z') - \phi(x,y,z) = \int_{(x,y,z)}^{(x',y',z')} \mathbf{F}\cdot d\mathbf{R} \tag{4.3.6}$$

If we use the definition of the derivative and (4.3.6)

$$\frac{\partial\phi}{\partial x} = \lim_{\triangle x\to 0} \frac{\phi(x+\triangle x, y, z) - \phi(x,y,z)}{\triangle x} = \lim_{\triangle x\to 0} \frac{1}{\triangle x}\int_{(x,y,z)}^{(x+\triangle x,y,z)} \mathbf{F}\cdot d\mathbf{R} \tag{4.3.7}$$

and express the resulting integral from (x,y,z) to $(x+\triangle x, y, z)$ as the integral along the the curve $(x+t, y, z)$, $0 \le t \le \triangle t$ we obtain

$$\frac{\partial\phi}{\partial x} = \lim_{\triangle x\to 0} \frac{1}{\triangle x}\int_x^{x+\triangle x} F_1(t,y,z)\, dt = F_1(x,y,z), \tag{4.3.8}$$

by the mean value theorem. The proof of that $\partial\phi/\partial y = F_2$ and $\partial\phi/\partial z = F_3$ is the same.

Ex. Show that $\mathbf{F} = xy^2\mathbf{i} + x^3y\mathbf{j}$ is not conservative. **Sol.** If $\partial\phi/\partial x = xy^2$ then $\partial^2\phi/\partial y\partial x = 2xy$ but if $\partial\phi/\partial y = x^3y$ then $\partial^2\phi/\partial x\partial y = 3x^2y$ which is a contradiction.