Solutions for Math 20E Midterm 1, Fall 98, Lindblad.

1. (a) $\mathbf{A} = \overline{PQ} = 2\mathbf{i} + 2\mathbf{k}$ and $\mathbf{B} = \overline{PR} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$ are parallel to the plane so $\mathbf{N} = \mathbf{A} \times \mathbf{B} = ... = -2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ is normal to the plane. The equation for the plane is $-2(x - -1) - 2(y-2) + 2(z-0) = 0$.

(b) The area of the triangle is $|\mathbf{A} \times \mathbf{B}|/2 = \sqrt{2^2 + 2^2 + 2^2}/2 = \sqrt{3}$.

2. We will need $\mathbf{R}'(t) = \frac{3}{2}(1+t)^{1/2}\mathbf{i} - \frac{3}{2}(1-t)^{1/2}\mathbf{j} + \mathbf{k}$.

$$(a) \quad \int_C ds = \int_0^1 |\mathbf{R}'(t)|\, dt = \int_0^1 \sqrt{\tfrac{9}{4}(1+t) + \tfrac{9}{4}(1-t) + 1}\, dt = \sqrt{\tfrac{11}{2}}.$$

$$(b) \quad \int_C \mathbf{F} \cdot d\mathbf{R} = \int_C y\, dx + x\, dy + x\, dz = \int_0^1 \left(y\frac{dx}{dt} + x\frac{dy}{dt} + x\frac{dz}{dt} \right) dt = ... =$$

$$= \int_0^1 \left((1-t)^{3/2}\tfrac{3}{2}(1+t)^{1/2} - (1+t)^{3/2}\tfrac{3}{2}(1-t)^{1/2} + (1+t)^{3/2} \right) dt$$

$$= \int_0^1 \left(-3t(1-t^2)^{1/2} + (1+t)^{3/2}\right) dt = (1-t^2)^{3/2} + \tfrac{2}{5}(1+t)^{5/2}\Big|_0^1 = -1 + \tfrac{2}{5}(2^{5/2} - 1)$$

3. (a) A flow line satisfy $\mathbf{R}'(t) = \beta(t)\mathbf{F}\big(\mathbf{R}(\mathbf{t})\big)$ for some scalar function β. i.e.

$$\frac{dx}{dt} = \beta F_1, \quad \frac{dy}{dt} = \beta F_2, \quad \frac{dz}{dt} = \beta F_3 \qquad \Longleftrightarrow \qquad \frac{dx}{F_1} = \frac{dy}{F_2} = \frac{dz}{F_3}$$

In our case

$$\frac{dx}{y} = \frac{dy}{x}, \quad dz = 0 \quad \Longrightarrow \quad x\, dx = y\, dy, \quad dz = 0 \quad \Longrightarrow \quad y^2 = x^2 + C_1, \quad z = C_2$$

Substituting in $(x,y,z) = (1,2,2)$ gives $C_1 = 3$, $C_2 = 2$ so $y = \sqrt{x^2+3}$ and $z = 2$.

(b) $\nabla \times \mathbf{F} = ... = 0$ and $\mathbf{F}$ is continuous everywhere so there is a potential.

(c) The potential satisfy
$$\begin{cases} \phi_x = & y \\ \phi_y = & x \\ \phi_z = & 0 \end{cases} \Longrightarrow \begin{cases} \phi = & xy + f(y,z) \\ \phi = & xy + g(y,z) \\ \phi = & h(x,y) \end{cases}$$

for any functions f, g and h. This has the solution $f = g = 0$ and $h = xy$, in which case $\phi(x,y,z) = xy$. It follows that $\int_C \mathbf{F} \cdot d\mathbf{R} = \phi(1,2,2) - \phi(0,0,0) = 2$.

4 (a) $\nabla \times \mathbf{F} = ... = 1/\sqrt{x^2+y^2}$. (b) $\nabla \cdot \mathbf{F} = ... = 0$. (c) No since, $\nabla \times \mathbf{F} \neq 0$.