MATH 20F LINEAR ALGEBRA

In Linear Algebra we solve systems of linear equations but linear algebra also provides a frame-work in which to think about many problems.

Lecture1: 1.1. A **linear system** of m **equations** in n **unknowns** is of the form:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n &= b_2 \\ &. \\ &. \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n &= b_m \end{aligned} \tag{1.1}$$

where the a_{ij}'s and b_i's are given constants and $x_1, x_2, ..., x_n$ are unknowns to be determined. It is called and $m \times n$ **system**. A **solution** to the system is an ordered n-tuple $(x_1, x_2, ..., x_n)$ such that all the m equations are satisfied. The set of all solutions are called the **solution set**.

Let us try to understand the geometric meaning of a general 2×2 **systems**:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$

The solutions to each of the equations form a line in the (x_1, x_2)-plane.
(x_1, x_2) is therefore a solution to the system if and only if it lies on both these lines.

Ex 1 Find all solutions to the system $\begin{cases} x_1 + x_2 = 3 \\ 2x_1 - x_2 = 0 \end{cases}$

Sol The two lines in the plane intersect at the point $(1, 2)$, which is the only solution

Ex 2 Find all solutions to the system $\begin{cases} x_1 + x_2 = 3 \\ 2x_1 + 2x_2 = 0 \end{cases}$

Sol The lines are parallel so they don't intersect. No solutions!

Ex 3 Find all solutions to the system: $\begin{cases} x_1 + x_2 = 3 \\ 2x_1 + 2x_2 = 6 \end{cases}$

Sol Both equations represent the same line. Every point on the line is a solution!

The same picture hold for a general system: the solution set is either empty (no solutions), or, there is exactly one solution, or there are infinitely many solutions.
A system is called **consistent** if it has at least one solution and **inconsistent** if it has no solutions.

Two systems are called **equivalent** if they have the same solution set.

Ex 4 Show that the systems are equivalent:

$$(I): \quad \begin{array}{r} x_1 + x_2 = 3 \\ 2x_1 - x_2 = 0 \end{array}, \qquad \Leftrightarrow \qquad (II): \quad \begin{array}{r} x_1 + x_2 = 3 \\ x_2 = 2 \end{array}$$

Sol Both systems represent the intersection of two lines that happen to intersect at the same point $(1, 2)$. This can also be seen analytically, in fact if we subtract 2 times the first line of the first system from the second line of the first system we get

$$\begin{array}{c} \text{[equation 2]} \\ -2\,\text{[equation 1]} \\ \hline \text{[new equation 2]} \end{array} \qquad \begin{array}{r} 2x_1 - x_2 = 0 \\ -2x_1 - 2x_2 = -6 \\ \hline -3x_2 = -6 \end{array}$$

If we divide both sides by -3 we get the second equation of the first system. Hence any solution to the first system is a solution to the second and going backwards we also see that any solution to the second system is a solution to the first.

The second system II can be solved analytically by so called **back-substitution**: If we plug $x_2 = 2$ into the first equation we get $2x_1 - 2 = 0$, i.e. $x_1 = 1$. To solve the first system I analytically we reduce it to the second system II and solve it.

If we follow the same procedure to solve the system in Ex 2 we get into trouble

$$\begin{array}{c} \text{[equation 2]} \\ -2\,\text{[equation 1]} \\ \hline \text{[new equation 2]} \end{array} \qquad \begin{array}{r} 2x_1 + 2x_2 = 0 \\ -2x_1 - 2x_2 = -6 \\ \hline 0 = -6 \end{array}$$

which is not true. Hence we analytically found that the system in Ex 2 is inconsistent.

An $n \times n$ system (1.1) is said to be in **non-degenerate triangular form** if $a_{ij} = 0$, for $i > j$, and $a_{ii} \neq 0$, for $i = 1, ..., n$. The entries a_{ii} for $i = 1, ..., n$ are called the **diagonal entries**. It is easy to solve a system in non-degenerate triangular form back-substitution. If possible we therefore want to transform $n \times n$ systems into equivalent non-degenerate triangular systems.

Let us recall what operations we can do that leads to equivalent systems:
I) We can change the order of any two equations
II) Both sides of an equation can be multiplied by the same nonzero number.
III) A multiple of one line may be added to another.

We want to solve the 3×3 system

$$\begin{array}{r} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{array}$$

To minimize the writing it is convenient to only write out the coefficients:

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right]$$

This is called the **augmented matrix**

Ex 5 Transform the system above into an equivalent system in non-degenerate triangular form.
Sol We want to eliminate x_1 from the last equation by using the first:

$$\begin{array}{c} \text{[equation 3]} \\ +4\,\text{[equation 1]} \\ \hline \text{[new equation 3]} \end{array} \qquad \begin{array}{r} -4x_1 + 5x_2 + 9x_3 = -9 \\ 4x_1 - 8x_2 + 4x_3 = 0 \\ \hline -3x_2 + 13x_3 = -9 \end{array}$$

After some practice this calculation is usually performed mentally.
Hence we get the system (written in both ways for comparison)

$$\begin{array}{r} x_1 - 2x_2 + \ \ x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -3x_2 + 13x_3 = -9 \end{array} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

Now first multiply the second equation by 1/2:

$$\begin{array}{r} x_1 - 2x_2 + \ \ x_3 = 0 \\ x_2 - 4x_3 = 4 \\ -3x_2 + 13x_3 = -9 \end{array} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

We now want to eliminate x_2 from the last equation by using the second:

$$\begin{array}{c} \text{[equation 3]} \\ +3\,\text{[equation 2]} \\ \hline \text{[new equation 3]} \end{array} \qquad \begin{array}{r} -3x_2 + 13x_3 = -9 \\ 3x_2 - 12x_3 = 12 \\ \hline x_3 = 3 \end{array}$$

Hence we get the equivalent system in non-degenerate triangular form:

$$\begin{array}{r} x_1 - 2x_2 + \ \ x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ x_3 = 3 \end{array} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Ex 6 Solve the system in Ex 5.
Sol Using the non-degenerate triangular form obtained in the previous example we can easily solve it using back-substitution. Two mental calculations are

$$\begin{array}{c} \text{[eq. 2]} \\ +4\,\text{[eq. 2]} \\ \hline \text{[new eq. 2]} \end{array} \qquad \begin{array}{r} x_2 - 4x_3x_3 = 4 \\ 4x_3 = 12 \\ \hline x_2 = 16 \end{array} \qquad \begin{array}{c} \text{[eq. 1]} \\ -1\,\text{[eq. 3]} \\ \hline \text{[new eq. 1]} \end{array} \qquad \begin{array}{r} x_1 - 2x_2 + x_3 = 0 \\ -x_3 = -3 \\ \hline x_1 - 2x_2 \qquad = -3 \end{array}$$

Hence we obtain

$$\begin{array}{r} x_1 - 2x_2 \qquad = -3 \\ x_2 \qquad = 16 \\ x_3 = 3 \end{array} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Now having cleared up the column above x_3 in equation 3, move back to the x_2 in equation 2 and use it to eliminate the $-2x_2$ above it. Adding 2 times the second equation to the first gives

$$\begin{array}{r} x_1 \qquad\qquad = 29 \\ x_2 \qquad = 16 \\ x_3 = 3 \end{array} \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$