Lecture 26: 6.3 Diagonalization. Recall that an $n \times n$ matrix is called diagonalizable if it can be written $A = XDX^{-1}$ where D is diagonal. We have seen that an $n \times n$ matrix is diagonalizable if and only if it has n linearly independent eigenvectors. We have seen that it has n linearly independent eigenvectors if it has n distinct eigenvalues. However, even if its characteristic polynomial has a multiple root it could still be that it has n linearly independent eigenvectors:

Ex 1 The matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is not diagonalizable but the matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is.

Sol Both have characteristic polynomial $p(\lambda) = (1-\lambda)^2$. We have

$$(A - 1 \cdot I)\mathbf{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad \begin{matrix} x_2 = 0 \\ 0 = 0 \end{matrix} \quad \Leftrightarrow \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Since it does not exist two linearly independent eigenvectors it can not be diagonalized. However, any vector is an eigenvector to the indentity matrix I and anyway it is already in diagonal form.

Ex 2 Diagonalize $A = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$

Sol 1 The eigenvalues and eigenvectors are $A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = - \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $A \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Hence $X^{-1}AX = D$, where $X = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $D = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$ and $X^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$.

Ex 3 Calculate the exponential matrix e^A for A in Ex 2.

Sol

$$e^A = e^{XDX^{-1}} = Xe^DX^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{-1} & 0 \\ 0 & e^3 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} = \begin{bmatrix} \dfrac{e^{-1} + e^3}{2} & \dfrac{e^{-1} - e^3}{2} \\ \dfrac{e^{-1} - e^3}{2} & \dfrac{e^{-1} + e^3}{2} \end{bmatrix}$$

Ex 4 Find the solution to the initial value problem:

$$\begin{aligned} x_1' &= x_1 - 2x_2 \qquad & x_1(0) &= 1 \\ x_2' &= -2x_1 + x_2, \qquad & x_2(0) &= 3 \end{aligned}$$

Sol

$$e^{At} = e^{XDtX^{-1}} = Xe^{Dt}X^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{3t} \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} = \begin{bmatrix} \dfrac{e^{-t} + e^{3t}}{2} & \dfrac{e^{-t} - e^{3t}}{2} \\ \dfrac{e^{-t} - e^{3t}}{2} & \dfrac{e^{-t} + e^{3t}}{2} \end{bmatrix}$$

and $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = e^{At} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2e^{-t} - e^{3t} \\ 2e^{-t} + e^{3t} \end{bmatrix}$

6.4 Spectral Theorem.

Spectral Th. If A is a real and symmetric $n \times n$ matrix then it has an orthonormal set of n eigenvectors and hence can be diagonalized by an orthogonal matrix Q. $Q^{-1}AQ = D$, where D is diagonal and $Q^{-1} = Q^T$.

Ex Diagonalize $\begin{bmatrix} 0 & 2 & -1 \\ 2 & 3 & -2 \\ -1 & -2 & 0 \end{bmatrix}$ with an orthogonal transformation.

Sol See book.