Math 20F Linear Algebra

In Linear Algebra we solve systems of linear equations but linear algebra also provides a frame-work in which to think about many problems.

Lecture 1: 1.1. A **linear system** of m **equations** in n **unknowns** is of the form:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n &= b_2 \\ &. \\ &. \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n &= b_m \end{aligned} \tag{1.1}$$

where the a_{ij}'s and b_i's are given constants and $x_1, x_2, ..., x_n$ are unknowns to be determined. It is called and $m \times n$ **system**. A **solution** to the system is an ordered n-tuple $(x_1, x_2, ..., x_n)$ such that all the m equations are satisfied. The set of all solutions are called the **solution set**.

Let us try to understand the geometric meaning of a general 2×2 **systems**:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$

The solutions to each of the equations form a line in the (x_1, x_2)-plane.
(x_1, x_2) is therefore a solution to the system if and only if it lies on both these lines.

Ex 1 Find all solutions to the system $\begin{cases} x_1 + x_2 = 3 \\ 2x_1 - x_2 = 0 \end{cases}$

Sol The two lines in the plane intersect at the point $(1, 2)$, which is the only solution

Ex 2 Find all solutions to the system $\begin{cases} x_1 + x_2 = 3 \\ 2x_1 + 2x_2 = 0 \end{cases}$

Sol The lines are parallel so they don't intersect. No solutions!

Ex 3 Find all solutions to the system: $\begin{cases} x_1 + x_2 = 3 \\ 2x_1 + 2x_2 = 6 \end{cases}$

Sol Both equations represent the same line. Every point on the line is a solution!

The same picture hold for a general system: the solution set is either empty (no solutions), or, there is exactly one solution, or there are infinitely many solutions.
A system is called **consistent** if it has at least one solution and **inconsistent** if it has no solutions.

Two systems are called **equivalent** if they have the same solution set.
Ex 4 Show that the systems are equivalent:

$$(I):\quad \begin{aligned} x_1 + x_2 &= 3 \\ 2x_1 - x_2 &= 0 \end{aligned}, \qquad \Leftrightarrow \qquad (II):\quad \begin{aligned} x_1 + x_2 &= 3 \\ x_2 &= 2 \end{aligned}$$

Sol Both systems represent the intersection of two lines that happen to intersect at the same point $(1, 2)$. This can also be seen analytically, in fact if we subtract 2 times the first line of the first system from the second line of the first system we get

$$\begin{array}{c} \text{[equation 2]} \\ -2\,\text{[equation 1]} \\ \hline \text{[new equation 2]} \end{array} \qquad \begin{array}{r} 2x_1 - x_2 = 0 \\ -2x_1 - 2x_2 = -6 \\ \hline -3x_2 = -6 \end{array}$$

If we divide both sides by -3 we get the second equation of the first system. Hence any solution to the first system is a solution to the second and going backwards we also see that any solution to the second system is a solution to the first.

The second system II can be solved analytically by so called **back-substitution**: If we plug $x_2 = 2$ into the first equation we get $2x_1 - 2 = 0$, i.e. $x_1 = 1$. To solve the first system I analytically we reduce it to the second system II and solve it.

If we follow the same procedure to solve the system in Ex 2 we get into trouble

$$\begin{array}{c} \text{[equation 2]} \\ -2\,\text{[equation 1]} \\ \hline \text{[new equation 2]} \end{array} \qquad \begin{array}{r} 2x_1 + 2x_2 = 0 \\ -2x_1 - 2x_2 = -6 \\ \hline 0 = -6 \end{array} \qquad \text{so we get the system} \qquad \begin{aligned} x_1 + x_2 &= 3 \\ 0 &= 6 \end{aligned}$$

which is not true. Hence we analytically found that the system in Ex 2 is inconsistent.

An $n \times n$ system (1.1) is said to be in **triangular form** if $a_{ij} = 0$, for $i > j$. The entries a_{ii} for $i = 1, ..., n$ are called the **diagonal entries**. A triangular system with nonvanishing diagonal entries is said to be **non-degenerate**.
It is easy to solve a system in non-degenerate triangular form by back-substitution.
We therefore want to transform $n \times n$ systems into equivalent triangular systems.
Let us recall what basic operations we can do that leads to equivalent systems:
1. A multiple of one equation may be added to another.
2. We can change the order of any two equations
3. Both sides of an equation can be multiplied by the same nonzero number.

We want to solve the 3×3 system

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned}$$

Geometrically this represents the intersection of 3 planes.
To minimize the writing it is convenient to only write out the **coefficient matrix**:

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix}, \qquad \text{and right hand side column vector} \qquad \begin{bmatrix} 0 \\ 8 \\ -9 \end{bmatrix}$$

or to combine them in one to the **augmented matrix** of the system.

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right] \qquad \text{or just} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}.$$

The first **row** of the matrix corresponds to the first equation etc.

Ex 5 Transform the system above into an equivalent triangular system and solve it.
Sol We want to eliminate x_1 from the last equation by using the first:

$$\begin{array}{c|c} \text{[equation 3]} & -4x_1 + 5x_2 + 9x_3 = -9 \\ +4\,\text{[equation 1]} & 4x_1 - 8x_2 + 4x_3 = 0 \\ \hline \text{[new equation 3]} & -3x_2 + 13x_3 = -9 \end{array}$$

After some practice this calculation is usually performed mentally.
Hence we get the system (written in both ways for comparison)

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -3x_2 + 13x_3 &= -9 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right] \; (3)+4(1)$$

Now first multiply the second equation by 1/2:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ -3x_2 + 13x_3 &= -9 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array}\right] \; (2)/2$$

We now want to eliminate x_2 from the last equation by adding 3 times the second:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 8 \\ x_3 &= 3 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right] \; (3)+3(1)$$

Hence we got an equivalent system in non-degenerate triangular form.
Because the diagonal entries are nonvanishing we can solve it using back substitution:

$$\begin{aligned} x_1 - 2x_2 &= -3 \\ x_2 &= 16 \\ x_3 &= 3 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right] \; \begin{array}{l} (1)-(3) \\ (2)+4(3) \\ \end{array}$$

Now having cleared up the column above x_3 in equation 3, move back to the x_2 in equation 2 and use it to eliminate the $-2x_2$ above it. Adding 2 times the second equation to the first gives

$$\begin{aligned} x_1 &= 29 \\ x_2 &= 16 \\ x_3 &= 3 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right] \; \begin{array}{l} (1)+2(2) \\ \\ \end{array}$$

Operations on the system corresponds to operations on the augmented matrix.
The **Elementary Row Operations** on a matrix are
1. A multiple of one row may be added to another.
2. Interchange two rows
3. Multiplied all entries in a row by the same nonzero number.
Two matrices are **row equivalent** if one can be transformed into the other by elementary row operations. Two systems have the same solution set if their augmented matrices are row equivalent.