Lecture 13: 4.4 Coordinate Systems.

Recall that $\mathbf{b}_1, ..., \mathbf{b}_n$ form a **basis** for a vector space V if
(i) $\mathbf{b}_1, ..., \mathbf{b}_n$ are linearly independent, and (ii) $\mathbf{b}_1, ..., \mathbf{b}_n$ span V.

The Unique Representation Theorem Let $\mathbf{b}_1, ..., \mathbf{b}_n$ be a basis for a vector space V. Then for any $\mathbf{x} \in V$, there is unique set of scalars $c_1, \dots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n. \tag{4.1}$$

Pf Since $\mathbf{b}_1, ..., \mathbf{b}_n$ span V there are scalars such that (4.1) holds.
If there was another set of scalars $d_1, \dots, d_n$ such that $\mathbf{x} = d_1\mathbf{b}_1 + \cdots c_n\mathbf{b}_n$ then

$$\mathbf{0} = \mathbf{x} - \mathbf{x} = (c_1 - d_1)\mathbf{b}_1 + \cdots (c_n - d_n)\mathbf{b}_n$$

and since $\mathbf{b}_1, ..., \mathbf{b}_n$ are linearly independent it follows that $c_1 - d_1 = \cdots = c_n - d_n = 0$.

Suppose $\mathcal{B} = \{\mathbf{b}_1, ..., \mathbf{b}_n\}$ is a basis. The **coordinates of x relative to the basis** $\mathcal{B}$ (or the $\mathcal{B}$**-coordinates of x**) are the weights $c_1, \cdots, c_n$ such that $\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$. The vector

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

is called the **coordinate vector of x (relative to** $\mathcal{B}$**)** or the $\mathcal{B}$**-coordinate vector of x**. The mapping $\mathbf{x} \to [\mathbf{x}]_{\mathcal{B}}$ is called the **coordinate mapping (determined by** $\mathcal{B}$**.)**

Ex Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$, where $\mathbf{b}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\mathbf{b}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and let $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2\}$, where $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Let $\mathbf{x} = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$. Find $[\mathbf{x}]_{\mathcal{B}}$ and $[\mathbf{x}]_{\mathcal{E}}$.
Sol We can write

$$\begin{bmatrix} 6 \\ 5 \end{bmatrix} = 6\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

so $[\mathbf{x}]_{\mathcal{E}} = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$. We want to find c_1 and c_2 such that

$$\begin{bmatrix} 6 \\ 5 \end{bmatrix} = c_1\begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

solving the system gives that $c_1 = 2$ and $c_2 = 3$. Hence $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

One can graph this in standard $\mathcal{E}$ graph paper and in $\mathcal{B}$ graph paper.

Note that in the example

$$\begin{bmatrix} 6 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

In general for a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$;

$$\mathbf{x} = P_{\mathcal{B}} [\mathbf{x}]_{\mathcal{B}}, \qquad \text{where} \qquad P_{\mathcal{B}} = [\,\mathbf{b}_1 \quad \cdots \quad \mathbf{b}_n\,]$$

and $[\mathbf{x}]_{\mathcal{B}}$ the coordinate vector. We call $P_{\mathcal{B}}$ the **change-of-coordinate matrix** from the standard basis in $\mathbf{R}^n$ to the basis $\mathcal{B}$.

Ex Find the coordinates of $\mathbf{x} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$ in the basis $\mathbf{b}_1, \mathbf{b}_2$ in the previous example.

Sol $P_{\mathcal{B}} = [\,\mathbf{b}_1 \quad \mathbf{b}_2\,] = \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix}$ and $P_{\mathcal{B}}^{-1} = \begin{bmatrix} 1/3 & 0 \\ -1/3 & 1 \end{bmatrix}$. Then $[\mathbf{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}\mathbf{x} = \begin{bmatrix} 1/3 & 0 \\ -1/3 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$.

The coordinate mapping allows us to introduce coordinate systems for unfamiliar vector spaces:

Ex Standard basis for the polynomials of degree 2 or less, $\mathbf{P}_2$ is $\{1, t, t^2\}$. We can write $[a + bt + ct^2]_{\mathcal{B}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. We say that $\mathbf{P}_2$ is isomorphic to $\mathbf{R}^3$. All vector space operations in $\mathbf{P}_2$ corresponds to operations in $\mathbf{R}^3$, e.g. adding two polynomials $(-1+2t-3t^2)+(2+3t+5t^2) = 1+5t+2t^2$ corresponds to adding their coordinate vectors: $\begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}$.

Th If $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis for a vector space V then the coordinate map $\mathbf{x} \to [\mathbf{x}]_{\mathcal{B}}$ is a one-to-one linear transformation from V to $\mathbf{R}^n$.

Assume that $\mathcal{B}$ is a basis. Show that the set $\mathbf{u}_1, \ldots, \mathbf{u}_p$ are linearly independent if and only if $\{[\mathbf{u}_1]_{\mathcal{B}}, \ldots, [\mathbf{u}_p]_{\mathcal{B}}\}$ are linearly independent.

Coordinate vectors allow us to associate vector spaces with subspaces:

Ex Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$, $\mathbf{b}_1 = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$ and $H = \text{Span}\{\mathbf{b}_1, \mathbf{b}_2\}$. Then $\mathbf{x} = \begin{bmatrix} 9 \\ 13 \\ 15 \end{bmatrix} \in H$. Find $[\mathbf{x}]_{\mathcal{B}}$.

Sol Find c_1 and c_2 such that $c_1 \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 13 \\ 15 \end{bmatrix}$. Augmented matrix

$$\begin{bmatrix} 3 & 0 & 9 \\ 3 & 1 & 13 \\ 1 & 3 & 15 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

so $c_1 = 3$ and $c_2 = 4$ and hence $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

Every vector on H is associated with a unique vector in $\mathbf{R}^2$. H is isomorphic to $\mathbf{R}^2$.

Draw the diagram of the net of $\mathbf{b}_1, \mathbf{b}_2$ on H.