

**Lecture 15: 3.1 Determinants.** As it turns out we can calculate a number called determinant for any given  $n \times n$  matrix that determines if the matrix is nonsingular.

**$2 \times 2$  matrices** By a previous theorem  $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$  is nonsingular if and only if it is row equivalent to the identity  $I$ . Assume first that  $a_{11} \neq 0$ .

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \sim (2) - (1)a_{21}/a_{11} \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} - a_{12}a_{21}/a_{11} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ 0 & \frac{a_{11}a_{22} - a_{21}a_{12}}{a_{11}} \end{bmatrix}$$

The resulting matrix is nonsingular if and only if  $a_{11}a_{22} - a_{21}a_{12} \neq 0$ . We define

$$(1) \quad \det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

Notice that this quantity changes sign but the magnitude stays the same if we switch the two rows. Since switching two rows leads to row equivalent systems it does not effect the invertibility of  $A$ . The nonvanishing of (1) in case  $a_{21} \neq 0$  is therefore also equivalent to that  $A$  is nonsingular. Furthermore if  $a_{11} = a_{21} = 0$  then the matrix clearly singular and determinant defined above also vanishes.

**$3 \times 3$  matrices** Let us first assume that  $a_{11} \neq 0$ .

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \sim (2) - (1)\frac{a_{21}}{a_{11}} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & \frac{a_{11}a_{22} - a_{21}a_{12}}{a_{11}} & \frac{a_{11}a_{23} - a_{21}a_{13}}{a_{11}} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \sim (3) - (1)\frac{a_{31}}{a_{11}} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & \frac{a_{11}a_{22} - a_{21}a_{12}}{a_{11}} & \frac{a_{11}a_{23} - a_{21}a_{13}}{a_{11}} \\ 0 & \frac{a_{11}a_{32} - a_{31}a_{12}}{a_{11}} & \frac{a_{11}a_{33} - a_{31}a_{13}}{a_{11}} \end{bmatrix} \sim$$

If we subtract  $\frac{a_{11}a_{32} - a_{31}a_{12}}{a_{11}a_{22} - a_{21}a_{12}}$  times the second row from the third we get

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & \frac{a_{11}a_{22} - a_{21}a_{12}}{a_{11}} & \frac{a_{11}a_{23} - a_{21}a_{13}}{a_{11}} \\ 0 & 0 & \frac{a_{11}a_{33} - a_{31}a_{13}}{a_{11}} - \frac{a_{11}a_{32} - a_{31}a_{12}}{a_{11}a_{22} - a_{21}a_{12}} \frac{a_{11}a_{23} - a_{21}a_{13}}{a_{11}} \end{bmatrix}$$

This matrix is now row equivalent to the identity if the diagonal elements are nonvanishing, i.e. if their product is nonvanishing:

$$\begin{aligned} & a_{11} \frac{a_{11}a_{22} - a_{21}a_{12}}{a_{11}} \left( \frac{a_{11}a_{33} - a_{31}a_{13}}{a_{11}} - \frac{a_{11}a_{32} - a_{31}a_{12}}{a_{11}a_{22} - a_{21}a_{12}} \frac{a_{11}a_{23} - a_{21}a_{13}}{a_{11}} \right) \\ &= \frac{1}{a_{11}} \left( (a_{11}a_{22} - a_{21}a_{12})(a_{11}a_{33} - a_{31}a_{13}) - (a_{11}a_{32} - a_{31}a_{12})(a_{11}a_{23} - a_{21}a_{13}) \right) \\ &= a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{12}a_{21}a_{33} + a_{12}a_{31}a_{23} + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22} \neq 0 \end{aligned}$$

We therefore define

$$\det(A) = a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{12}a_{21}a_{33} + a_{12}a_{31}a_{23} + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22}$$

Since this only changes sign if we switch two rows and switching two rows does not effect the invertibility we see that the same result is true if any entry in the first column is nonvanishing. But if all vanish the matrix is not invertible and  $\det(A) = 0$ .

Notice now that we can write the determinant of a  $3 \times 3$  matrix in terms of  $2 \times 2$  ones:

$$\begin{aligned} \det(A) &= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22}) \\ &= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ &= a_{11} \begin{bmatrix} & a_{22} & a_{23} \\ & a_{32} & a_{33} \end{bmatrix} - a_{12} \begin{bmatrix} a_{21} & & a_{23} \\ a_{31} & & a_{33} \end{bmatrix} + a_{13} \begin{bmatrix} a_{21} & a_{22} & \\ a_{31} & a_{32} & \end{bmatrix} \end{aligned}$$

This can be interpreted as  $a_{11}$  times the determinant of the  $2 \times 2$  matrix obtained from  $A$  by deleting the row and column of  $a_{11}$  minus  $a_{12}$  times the determinant of the  $2 \times 2$  matrix obtained by deleting the row and column of  $a_{12}$  plus  $a_{13}$  times the determinant of the  $2 \times 2$  matrix obtained by deleting the row and column of  $a_{13}$ .

**Ex** Find the determinant of  $A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$ .

**Sol**

$$\det(A) = 1 \begin{vmatrix} 4 & -1 \\ -2 & 0 \end{vmatrix} - 5 \begin{vmatrix} 2 & -1 \\ 0 & 0 \end{vmatrix} + 0 \begin{vmatrix} 2 & -4 \\ 0 & -2 \end{vmatrix} = -2 - 0 + 0 = -2$$

**Def** Let  $A = (a_{ij})$  be an  $n \times n$  matrix and let  $A_{ij}$  denote the  $(n-1) \times (n-1)$  matrix obtained from  $A$  by deleting the row and column containing  $a_{ij}$ . The **cofactors** are

$$C_{ij} = (-1)^{i+j} \det(A_{ij})$$

and the **determinant** is

$$(2) \quad \det A = a_{11}C_{11} + \dots + a_{1n}C_{1n}.$$

The definition is recursive; it assumes that we already defined the  $(n-1) \times (n-1)$  determinants, but we can use the definition repeatedly to reduce to  $2 \times 2$  determinants that we defined by (1), or we can reduce to  $1 \times 1$  determinants  $\det(a) = a$ .

The signs above are determined by

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdot \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

It turns out that in fact we can expand along any row or column:

**Th** For any  $i$  and  $j$  we have

$$\det A = a_{i1}C_{i1} + \dots + a_{in}C_{in} = a_{1j}C_{1j} + \dots + a_{nj}C_{nj}$$

**Ex** Find the determinant below. **Sol** Expanding along the first column;

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 5 \end{vmatrix} = 1 \begin{vmatrix} 2 & 1 & 5 \\ 0 & 2 & 1 \\ 0 & 3 & 5 \end{vmatrix} - 0 + 0 - 0 = 1 \cdot 2 \begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix} = 2(2 \cdot 5 - 1 \cdot 3) = 14.$$

**Th** If  $A$  is triangular the determinant is equal to the product of the diagonal elements.

$$\mathbf{Ex} \quad \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 5 \end{vmatrix} = 1 \cdot 2 \cdot 2 \cdot 5 = 20$$