Lecture 19: 5.3 Diagonalization.
We will show that some square matrices A can be factorized $A = PDP^{-1}$, where D is diagonal (i.e. the entries off the main diagonal are all zeros).
This can be used to compute A^k, for large k, which is useful in the applications.
(If multiplying by A represents the evolution of a system during one time unit then multiplying by A^k represents the evolution after k time units)

Ex Let $D = \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix}$. Compute D^2, D^3 and D^k.
Sol $D^2 = \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 5^2 & 0 \\ 0 & 4^2 \end{bmatrix}$, $D^3 = DD^2 = \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 5^2 & 0 \\ 0 & 4^2 \end{bmatrix} = \begin{bmatrix} 5^3 & 0 \\ 0 & 4^3 \end{bmatrix}$,
$D^k = \begin{bmatrix} 5^k & 0 \\ 0 & 4^k \end{bmatrix}$

Ex Let $A = \begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix}$. Compute A^k.
Use that $A = PDP^{-1}$, where $D = \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, $P^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$.
Sol We have $A^2 = PDP^{-1}PDP^{-1} = PDIDP^{-1} = PD^2P^{-1}$,

$$A^k = PD^kP^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 5^k & 0 \\ 0 & 4^k \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2\cdot 5^k - 4^k & -5^k + 4^k \\ 2\cdot 5^k - 2\cdot 4^k & -5^k + 2\cdot 4^k \end{bmatrix}$$

A square matrix A is called **diagonalizable** if it can be written $A = PDP^{-1}$, where D is diagonal and P is invertible.
When is A diagonalizable and if it is how do we find D and P?
The answer lies in the eigenvalues and eigenvectors.
Note that

$$\begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

so the columns of P are made out of the eigenvectors of A and the diagonal entries of D are the eigenvalues of A. We can put this to equations together in one matrix equation:

$$\begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 5 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix},$$

i.e.

$$\begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1},$$

In general if A is an $n\times n$ matrix with n linearly independent eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ and eigenvalues $\lambda_1, \dots, \lambda_n$ then

$$A\,[\,\mathbf{v}_1 \ \cdots \ \mathbf{v}_n\,] = [\,A\mathbf{v}_1 \ \cdots \ A\mathbf{v}_n\,] = [\,\lambda_1\mathbf{v}_1 \ \cdots \ \lambda_n\mathbf{v}_n\,] = [\,\mathbf{v}_1 \ \cdots \ \mathbf{v}_n\,]\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

and hence

$$A = [\,\mathbf{v}_1 \ \cdots \ \mathbf{v}_n\,]\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}[\,\mathbf{v}_1 \ \cdots \ \mathbf{v}_n\,]^{-1}$$

Diagonalization Theorem An $n \times n$ matrix is diagonalizable A if and only if it has n linearly independent eigenvectors.

Ex If possible, diagonalize $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$.

Sol The eigenvalues $\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 0 & 0 \\ 1 & 2-\lambda & 1 \\ -1 & 0 & 1-\lambda \end{vmatrix} = (2-\lambda)^2(1-\lambda) = 0.$

Basis for $\lambda = 1$: $\mathbf{v}_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$.

Basis for $\lambda = 2$: $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

Construct $P = [\,\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3\,] = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. $A = PDP^{-1}$.

Ex If possible, diagonalize $A = \begin{bmatrix} 2 & 4 & 6 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{bmatrix}$.

Sol The eigenvalues $\det(A - \lambda I) = (\lambda - 2)^2(\lambda - 4) = 0.$

Basis for $\lambda = 2$: $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Basis for $\lambda = 4$: $\mathbf{v}_2 = \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}$.

There are not three linearly independent eigenvectors so A can not be diagonalized.

Th If $\lambda_1, \ldots, \lambda_n$ are distinct eigenvalues of an $n \times n$ matrix A with corresponding eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$, then $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent.

Th If A is symmetric matrix $A^T = A$ then A has n linearly independent Eigenvectors.