Lecture 20: 5.4-5.6.

5.4 Expressing a linear transformation in terms of different bases.

Ex 2 Let L be the line in $\mathbf{R}^2$ that is spanned by the vector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.
Let T be the linear transformation that projects any vector orthogonally onto L. Find the matrix A for T in the standard coordinate system.

Sol Since the projection leaves the line invariant the vector $\mathbf{x}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ must be an eigenvector with eigenvalue 1: $A\mathbf{x}_1 = \mathbf{x}_1$. Moreover, since the orthogonal vector $\mathbf{x}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ is mapped to $\mathbf{0}$ its also an eigenvector with eigenvalue 0: $A\mathbf{x}_2 = \mathbf{0} = 0\mathbf{x}_2$.
If we express If $\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2$, in terms of the basis of eigenvectors then $A\mathbf{x} = c_1\mathbf{x}_1$.
Change of coordinates $\mathbf{x} = P\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, where $P = [\,\mathbf{x}_1\ \mathbf{x}_2\,] = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$, and $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = P^{-1}\mathbf{x}$, where $P^{-1} = \frac{1}{10}\begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}$.
Hence $A\mathbf{x} = c_1\mathbf{x}_1 = [\,\mathbf{x}_1\ \ \mathbf{x}_2\,]\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = PDP^{-1}$, where $D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.
The matrix D for T in the $\mathcal{B} = \{\mathbf{x}_1, \mathbf{x}_2\}$ coordinate system is hence very simple. The matrix for A for T in the standard coordinates is more complicated. The following diagram commute

$$
\begin{array}{ccc}
c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = \mathbf{x} & \xrightarrow{A} & A\mathbf{x} = c_1\mathbf{x}_1 \\
{\scriptstyle P}\uparrow & & \uparrow{\scriptstyle P} \\
\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = [\mathbf{x}]_{\mathcal{B}} & \xrightarrow{D} & [A\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ 0 \end{bmatrix}
\end{array},
$$

Hence

$$A = PDP^{-1} = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\frac{1}{10}\begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}$$

5.5 Complex Eigenvalues.

Ex 3 Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$.

Sol The eigenvalues are and eigenvectors are complex:
$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 1^2 = (1-\lambda-i)(1-\lambda+i) = 0,$
so the matrix can not be diagonalized. This is the matrix for a rotation with scaling:
$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \sqrt{2}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \sqrt{2}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, where $\theta = \pi/4$.

Other examples of matrices with complex eigenvalues are obtained by making a change of basis: $B = PAP^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ -5 & 4 \end{bmatrix}$.

5.6 Discrete Dynamical Systems.

Ex Denote the owl and rat population at time k by $\mathbf{x}_k = \begin{bmatrix} O_k \\ R_k \end{bmatrix}$. Suppose

$$\begin{aligned} O_{k+1} &= 0.5\, O_k + 0.4\, R_k \\ R_{k+1} &= -p\, O_k + 1.1\, R_k \end{aligned}$$

where $p = 0.104$, or $\mathbf{x}_{k+1} = A\mathbf{x}_k$, where $A = \begin{bmatrix} 0.5 & 0.4 \\ -.104 & 1.1 \end{bmatrix}$. The eigenvalues for the matrix A are $\lambda_1 = 1.02$ and $\lambda_2 = 0.58$ and the eigenvectors are $\mathbf{v}_1 = \begin{bmatrix} 10 \\ 13 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$. An initial $\mathbf{x}_0$ can be written $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$. Then for $k \geq 0$

$$\mathbf{x}_k = c_1 A^k \mathbf{v}_1 + c_2 A^k \mathbf{v}_2 = c_1\lambda_1^k \mathbf{v}_1 + c_2\lambda_2^k \mathbf{v}_2 = c_1(1.02)^k \begin{bmatrix} 10 \\ 13 \end{bmatrix} + c_2(0.58)^k \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

As k becomes large the first state will dominate and the other will go to $\mathbf{0}$ unless the initial conditions are such that $c_1 = 0$ in which case the whole solution goes to $\mathbf{0}$.