

Lecture 21: 6.1 Inner product, Length and Orthogonality.

The **inner product** or **dot product** between two vectors $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$ is

$$\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^T \mathbf{y} = [x_1 \ \cdots \ x_n] \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = x_1 y_1 + \cdots + x_n y_n.$$

The **length** of the vector \mathbf{x} is $\|\mathbf{x}\| = \sqrt{\mathbf{x} \cdot \mathbf{x}} = \sqrt{x_1^2 + \cdots + x_n^2}$

We know this is the Euclidean length in two dimensions by the Pythagorean law.

The **distance** between \mathbf{x} and \mathbf{y} is $\text{dist}(\mathbf{x}, \mathbf{y}) = \|\mathbf{y} - \mathbf{x}\|$.

Two vectors \mathbf{x} and \mathbf{y} are called **orthogonal** if $\mathbf{x} \cdot \mathbf{y} = 0$.

If \mathbf{x} is perpendicular to \mathbf{y} then it is perpendicular to $-\mathbf{y}$ and $\text{dist}(\mathbf{x}, \mathbf{y}) = \text{dist}(\mathbf{x}, -\mathbf{y})$:

$$\text{dist}(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|^2 = (\mathbf{x} - \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}) = \mathbf{x} \cdot \mathbf{x} + \mathbf{y} \cdot \mathbf{y} - \mathbf{x} \cdot \mathbf{y} - \mathbf{y} \cdot \mathbf{x} = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - 2\mathbf{x} \cdot \mathbf{y},$$

$$\text{dist}(\mathbf{x}, -\mathbf{y}) = \|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + 2\mathbf{x} \cdot \mathbf{y}.$$

It follows that $\mathbf{x} \cdot \mathbf{y} = 0$ if \mathbf{x} and \mathbf{y} are perpendicular so it is the same as orthogonal.

The **Pythagorean law**: $\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 = \|\mathbf{x} - \mathbf{y}\|^2$ if and only if $\mathbf{x} \cdot \mathbf{y} = 0$.

\mathbf{z} is said to be **orthogonal to** a subspace W if it is orthogonal to every vector in W .

The set of all vectors \mathbf{z} that are orthogonal to a subspace $W \subset \mathbf{R}^n$ is called the **orthogonal complement** of W and is denoted by W^\perp .

$$W^\perp = \{\mathbf{z} \in \mathbf{R}^n; \mathbf{z} \cdot \mathbf{y} = 0, \text{ for every } \mathbf{y} \in W\}$$

Ex If W is plane through the origin in \mathbf{R}^3 and L is the line through the origin perpendicular to W , then $W^\perp = L$. In fact, clearly $L \subset W^\perp$ since L is perpendicular to W and any vector not in L is not perpendicular to W . Similarly $L^\perp = W$.

Ex If $V = \{\mathbf{x} \in \mathbf{R}^3; \mathbf{x} = \alpha(1, 1, 1), \text{ for some } \alpha\}$ then $V^\perp = \{\mathbf{y} \in \mathbf{R}^3; \alpha(1, 1, 1) \cdot \mathbf{y} = 0, \text{ for every } \alpha\} = \{\mathbf{y} \in \mathbf{R}^3; y_1 + y_2 + y_3 = 0\}$.

(1) If $W = \text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ then $\mathbf{z} \in W^\perp$ if and only if $\mathbf{z} \cdot \mathbf{v}_1 = \cdots = \mathbf{z} \cdot \mathbf{v}_k = 0$.

(2) W^\perp is a subspace.

Th Let A be an $m \times n$ matrix. Then $(\text{Row } A)^\perp = \text{Nul } A$ and $(\text{Col } A)^\perp = \text{Nul } A^T$.

Pf If $\mathbf{x} \in \text{Nul } A$ then

$$\begin{array}{l} \mathbf{r}_1 \rightarrow \\ \mathbf{r}_2 \rightarrow \\ \vdots \\ \mathbf{r}_m \rightarrow \end{array} \begin{bmatrix} * & * & \cdots & * \\ * & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & * \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad A\mathbf{x} = \begin{bmatrix} \mathbf{r}_1 \cdot \mathbf{x} \\ \mathbf{r}_2 \cdot \mathbf{x} \\ \vdots \\ \mathbf{r}_m \cdot \mathbf{x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

so \mathbf{x} is orthogonal to $\text{Row } A$ since its orthogonal to $\mathbf{r}_1, \dots, \mathbf{r}_m$.

Ex Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \end{bmatrix}$. Basis for $\text{Nul } A = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, so $\text{Nul } A$ is a plane in

\mathbf{R}^3 . Basis for $\text{Row } A = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ so $\text{Row } A$ is a line in \mathbf{R}^3 . Basis for $\text{Col } A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, so

$\text{Col } A$ is a line in \mathbf{R}^2 . Basis for $\text{Nul } A^T = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, so $\text{Nul } A^T$ is a line in \mathbf{R}^2 .

6.2 Orthogonal Sets.

A set of vectors $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ is called an **orthogonal set** if $\mathbf{u}_i \cdot \mathbf{u}_j = 0$ when $i \neq j$.

Ex Show that $\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is an orthogonal set? **Sol:**

$$\mathbf{u}_1 \cdot \mathbf{u}_2 = 1 \cdot 1 + (-1) \cdot 1 + 0 \cdot 0 = 0, \quad \mathbf{u}_1 \cdot \mathbf{u}_3 = 1 \cdot 0 + (-1) \cdot 0 + 0 \cdot 1 = 0, \quad \mathbf{u}_2 \cdot \mathbf{u}_3 = 1 \cdot 0 + 1 \cdot 0 + 0 \cdot 1 = 0$$

Th Suppose that $S = \{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ is an orthogonal set of nonzero vectors in \mathbf{R}^n and $W = \text{Span}\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$. Then S is a linearly independent set and a basis for W .

Pf Suppose

$$c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 + \dots + c_p \mathbf{u}_p = \mathbf{0},$$

$$(c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 + \dots + c_p \mathbf{u}_p) \cdot \mathbf{u}_1 = 0,$$

$$c_1 \mathbf{u}_1 \cdot \mathbf{u}_1 + c_2 \mathbf{u}_2 \cdot \mathbf{u}_1 + \dots + c_p \mathbf{u}_p \cdot \mathbf{u}_1 = 0,$$

$$c_1 \mathbf{u}_1 \cdot \mathbf{u}_1 = 0,$$

$c_1 = 0$ since $\mathbf{u}_1 \cdot \mathbf{u}_1 > 0$. Similarly $c_2 = \dots = c_p = 0$, so S is a linearly independent set.