

Lecture 22: 6.2 cont.

An **orthogonal basis** $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ for a subspace W is a basis that is also orthogonal, i.e. $\mathbf{u}_i \cdot \mathbf{u}_j = 0$, whenever $i \neq j$.

Th If $S = \{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ is an orthogonal basis for a subspace W and $\mathbf{y} \in W$, then

$$\mathbf{y} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \dots + c_p\mathbf{u}_p, \quad \text{where } c_i = \frac{\mathbf{y} \cdot \mathbf{u}_i}{\mathbf{u}_i \cdot \mathbf{u}_i}$$

Pf

$$\mathbf{y} \cdot \mathbf{u}_1 = (c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \dots + c_p\mathbf{u}_p) \cdot \mathbf{u}_1 = c_1\mathbf{u}_1 \cdot \mathbf{u}_1 + c_2\mathbf{u}_2 \cdot \mathbf{u}_1 + \dots + c_p\mathbf{u}_p \cdot \mathbf{u}_1 = c_1\mathbf{u}_1 \cdot \mathbf{u}_1$$

$$\text{Hence } c_1 = \frac{\mathbf{u}_1 \cdot \mathbf{y}}{\mathbf{u}_1 \cdot \mathbf{u}_1} \text{ and similarly } c_2 = \frac{\mathbf{u}_2 \cdot \mathbf{y}}{\mathbf{u}_2 \cdot \mathbf{u}_2}, \quad \dots \quad c_p = \frac{\mathbf{u}_p \cdot \mathbf{y}}{\mathbf{u}_p \cdot \mathbf{u}_p}$$

Ex Write $\mathbf{y} = \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$ as a linear combination of $\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Sol Since $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal basis it follows from the previous theorem that $\mathbf{y} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3$, where

$$c_1 = \frac{\mathbf{u}_1 \cdot \mathbf{y}}{\mathbf{u}_1 \cdot \mathbf{u}_1} = -2, \quad c_2 = \frac{\mathbf{u}_2 \cdot \mathbf{y}}{\mathbf{u}_2 \cdot \mathbf{u}_2} = 5, \quad c_3 = \frac{\mathbf{u}_3 \cdot \mathbf{y}}{\mathbf{u}_3 \cdot \mathbf{u}_3} = 4$$

Hence $\mathbf{y} = -2\mathbf{u}_1 + 5\mathbf{u}_2 + 4\mathbf{u}_3$.

We will now calculate the **orthogonal projection** of \mathbf{y} onto \mathbf{u} .

It is a vector $\hat{\mathbf{y}} = \alpha\mathbf{u}$ in the direction of \mathbf{u} , such that $\mathbf{y} - \hat{\mathbf{y}}$ is orthogonal to \mathbf{u} :

$$(\mathbf{y} - \alpha\mathbf{u}) \cdot \mathbf{u} = 0 \quad \Leftrightarrow \quad \mathbf{y} \cdot \mathbf{u} - \alpha\mathbf{u} \cdot \mathbf{u} = 0 \quad \Leftrightarrow \quad \alpha = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}$$

The **orthogonal projection of \mathbf{y} onto \mathbf{u}** is the vector $\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u}$. We can write

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$$

where \mathbf{z} is called the **component orthogonal to \mathbf{u}** .

Ex Let $\mathbf{y} = \begin{bmatrix} -8 \\ 4 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Find the orthogonal projection $\hat{\mathbf{y}}$ of \mathbf{y} onto \mathbf{u} .

Write \mathbf{y} as a vector parallel to \mathbf{u} and a vector perpendicular to \mathbf{u} .

Compute the distance from \mathbf{y} to the line through $\mathbf{0}$ and \mathbf{u} .

$$\text{Sol } \hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}} = \frac{-24 + 4}{10} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ -2 \end{bmatrix}. \quad \mathbf{y} = \hat{\mathbf{y}} + (\mathbf{y} - \hat{\mathbf{y}}) \quad \text{We have } \mathbf{y} - \hat{\mathbf{y}} = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$$

$$\text{so } \|\mathbf{y} - \hat{\mathbf{y}}\| = \sqrt{2^2 + 6^2} = \sqrt{40}.$$

A set of vectors $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ is called **orthonormal** if it is an orthogonal set of unit vectors i.e.

$$\mathbf{u}_i \cdot \mathbf{u}_j = \delta_{ij} = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases}$$

If $W = \text{Span}\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ then $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ is an **orthonormal basis** for W .

Recall that \mathbf{v} is a unit vector if $\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = 1$.

Suppose that $U = [\mathbf{u}_1 \ \mathbf{u}_2 \ \mathbf{u}_3]$, where $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal set. Then

$$U^T U = \begin{bmatrix} \mathbf{u}_1^T \\ \mathbf{u}_2^T \\ \mathbf{u}_3^T \end{bmatrix} [\mathbf{u}_1 \ \mathbf{u}_2 \ \mathbf{u}_3] = \begin{bmatrix} \mathbf{u}_1^T \cdot \mathbf{u}_1 & \mathbf{u}_1^T \cdot \mathbf{u}_2 & \mathbf{u}_1^T \cdot \mathbf{u}_3 \\ \mathbf{u}_2^T \cdot \mathbf{u}_1 & \mathbf{u}_2^T \cdot \mathbf{u}_2 & \mathbf{u}_2^T \cdot \mathbf{u}_3 \\ \mathbf{u}_3^T \cdot \mathbf{u}_1 & \mathbf{u}_3^T \cdot \mathbf{u}_2 & \mathbf{u}_3^T \cdot \mathbf{u}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

A matrix such that $U^T U = I$ is called **orthogonal** (not orthonormal)

U is orthogonal if and only if the columns are orthonormal.

We have that multiplying by an orthogonal matrix preserves lengths and angles between vectors since $(U\mathbf{x}) \cdot (U\mathbf{y}) = (U\mathbf{x})^T U\mathbf{y} = \mathbf{x}^T U^T U\mathbf{y} = \mathbf{x}^T I\mathbf{y} = \mathbf{x}^T \mathbf{y} = \mathbf{x} \cdot \mathbf{y}$

6.3 Orthogonal sets.

Ex Suppose that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal basis for \mathbf{R}^3 and let $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$. Write $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$, where $\mathbf{y} \in W$ and $\mathbf{z} \in W^\perp$, i.e. \mathbf{z} is perpendicular to every vector in W , i.e. $\mathbf{z} \cdot \mathbf{u}_1 = 0 = \mathbf{z} \cdot \mathbf{u}_2$.

Sol By a previous theorem we can write $\mathbf{y} = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 + \frac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2} \mathbf{u}_2 + \frac{\mathbf{y} \cdot \mathbf{u}_3}{\mathbf{u}_3 \cdot \mathbf{u}_3} \mathbf{u}_3$

Let $\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 + \frac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2} \mathbf{u}_2$. Then $\hat{\mathbf{y}} \in W$ and $\mathbf{z} = \frac{\mathbf{y} \cdot \mathbf{u}_3}{\mathbf{u}_3 \cdot \mathbf{u}_3} \mathbf{u}_3$ is orthogonal to W , since $\mathbf{z} \cdot \mathbf{u}_1 = \mathbf{z} \cdot \mathbf{u}_2 = 0$.

$\hat{\mathbf{y}}$ is called the **projection of \mathbf{y} onto W**

Ex Let $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$, where $\mathbf{u}_1 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and let $\mathbf{y} = \begin{bmatrix} 0 \\ 3 \\ 10 \end{bmatrix}$.

Write $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$, where $\hat{\mathbf{y}} \in W$ and $\mathbf{z} \in W^\perp$.

Sol Since $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal basis it follows that

$$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} + \frac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2} = \frac{10}{10} \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + \frac{3}{1} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 9 \end{bmatrix}.$$

$$\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}} = \begin{bmatrix} 0 \\ 3 \\ 10 \end{bmatrix} - \begin{bmatrix} -3 \\ 0 \\ 9 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$$