Lecture 25: 6.7-6.8: Inner products and Fourier series.

Def An **inner product** on a vector space is a function that for each pair of vectors gives a real number: $V \ni \mathbf{x}, \mathbf{y} \to \langle \mathbf{x}, \mathbf{y} \rangle \in \mathbf{R}$, satisfying:
(i) $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$ with equality if and only if $\mathbf{x} = 0$.
(ii) $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$.
(iii) $\langle \alpha\mathbf{x} + \beta\mathbf{y}, \mathbf{z} \rangle = \alpha\langle \mathbf{x}, \mathbf{z} \rangle + \beta\langle \mathbf{y}, \mathbf{z} \rangle$.

Ex 1 $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$ and $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T\mathbf{y} = x_1y_1 + \cdots + x_ny_n$.

Ex 2 $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$ and $\langle \mathbf{x}, \mathbf{y} \rangle = x_1y_1w_1 + \cdots + x_ny_nw_n$, where $w_i > 0$, for $i = 1, ..., n$.

Ex 3 $f, g \in C[a, b]$, the continuous functions on the interval $[a, b]$, and

$$\langle f, g \rangle = \int_a^b f(x)g(x)\, dx$$

Def $\mathbf{x}$ and $\mathbf{y}$ are orthogonal if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$.

Def The norm is $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$.

Note that Phytagorean law holds: $\|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$, if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$.
This follows since $\|\mathbf{x}+\mathbf{y}\|^2 = \langle \mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{x} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle + 2\langle \mathbf{x}, \mathbf{y} \rangle = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$.

Ex 4 $V = C[0, 2\pi]$ is a vector space. Let W be the subspace spanned by all trigonometric polynomials up to order n: $1, \cos t, \dots, \cos nt, \sin t, \dots, \sin nt$, i.e. W consists of all functions of the form

$$\frac{a_0}{2} + a_1 \cos t + \cdots + a_n \cos nt + b_1 \sin t + \cdots + b_n \sin nt$$

The basis vectors $1, \cos t, \dots, \cos nt, \sin t, \dots, \sin nt$, are orthogonal to each other, i.e.

$$\int_0^{2\pi} \cos mt \sin nt\, dt = 0$$

$$\int_0^{2\pi} \cos mt \cos nt\, dt = 0, \qquad \text{if } m \neq n, \qquad = \pi, \text{if } m = n,$$

$$\int_0^{2\pi} \sin mt \sin nt\, dt = 0, \qquad \text{if } m \neq n \qquad = \pi, \text{if } m = n,$$

Using Euler's formulas, $\cos mt = \dfrac{e^{imt} + e^{-imt}}{2}$, $\sin mt = \dfrac{e^{imt} - e^{-imt}}{2i}$, the proof reduces to

$$\int_0^{2\pi} e^{imt}\, dt = 0, \qquad \text{if } m \neq 0, \qquad = \pi, \text{if } m = 0,$$

Question Given $f \in C[0, 2\pi]$ which is the function $p \in W$ closest to f, i.e. such that $\|f - p\|$ is as small as possible?

Answer The orthogonal projection of f onto W.

The orthogonal decomposition theorem Let W be a subspace of a vector space V and suppose that $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ is an orthogonal basis for W. Any $\mathbf{y} \in V$ can be written uniquely as

$$\mathbf{y} = \widehat{\mathbf{y}} + \mathbf{z},$$

where

$$\widehat{\mathbf{y}} = \frac{\langle \mathbf{y}, \mathbf{u}_1 \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle}\mathbf{u}_1 + \cdots + \frac{\langle \mathbf{y}, \mathbf{u}_p \rangle}{\langle \mathbf{u}_p, \mathbf{u}_p \rangle}\mathbf{u}_p$$

and $\mathbf{z} = \mathbf{y} - \widehat{\mathbf{y}} \in W^\perp$, the orthogonal complement $W^\perp = \{\mathbf{z} \in \mathbf{R}^n ; \langle \mathbf{z}, \mathbf{u}_1 \rangle = 0, \dots, \langle \mathbf{z}, \mathbf{u}_p \rangle = 0\}$. $\widehat{\mathbf{y}} = \mathrm{proj}_W \mathbf{y}$ is called the **orthogonal projection of y onto** W.

The best approximation theorem Let W be a subspace of a vector space V, $\mathbf{y}$ a vector and $\widehat{\mathbf{y}}$ be the orthogonal projection of $\mathbf{y}$ onto W. Then $\widehat{\mathbf{y}}$ is the point in W closest to $\mathbf{y}$:

$$||\mathbf{y} - \widehat{\mathbf{y}}|| < ||\mathbf{y} - \mathbf{v}||, \qquad \mathbf{v} \in W, \quad \mathbf{v} \neq \widehat{\mathbf{y}}.$$

Pf We can write

$$\mathbf{y} - \mathbf{v} = \mathbf{y} - \widehat{\mathbf{y}} + \widehat{\mathbf{y}} - \mathbf{v}$$

where $\mathbf{y} - \widehat{\mathbf{y}} \in W^\perp$ and $\widehat{\mathbf{y}} - \mathbf{v} \in W$ are orthogonal and hence by the Pythagorean theorem:

$$||\mathbf{y} - \mathbf{v}||^2 = ||\mathbf{y} - \widehat{\mathbf{y}}||^2 + ||\widehat{\mathbf{y}} - \mathbf{v}||^2 > ||\mathbf{y} - \widehat{\mathbf{y}}||^2.$$

The orthogonal projection of f onto W is given by

$$\widehat{\mathbf{f}} = \frac{\langle \mathbf{f}, 1 \rangle}{\langle 1, 1 \rangle}1 + \frac{\langle \mathbf{f}, \cos t \rangle}{\langle \cos t, \cos t \rangle}\cos t + \cdots + \frac{\langle \mathbf{f}, \cos nt \rangle}{\langle \cos nt, \cos nt \rangle}\cos nt$$
$$+ \frac{\langle \mathbf{f}, \sin t \rangle}{\langle \sin t, \sin t \rangle}\sin t + \cdots + \frac{\langle \mathbf{f}, \sin nt \rangle}{\langle \sin nt, \sin nt \rangle}\sin nt$$