

Lecture 26: 7.1 Diagonalization of symmetric matrices.

Spectral Th. If A is a real and symmetric $n \times n$ matrix then it has an orthonormal set of n eigenvectors and hence can be diagonalized by an orthogonal matrix U . $U^{-1}AU = D$, or $A = UDU^{-1}$, where D is diagonal and $U^{-1} = U^T$.

Ex Diagonalize $A = \begin{bmatrix} 0 & 2 & -1 \\ 2 & 3 & -2 \\ -1 & -2 & 0 \end{bmatrix}$ with an orthogonal transformation.

Sol A is symmetric so it can be diagonalized by an orthogonal transformation. The characteristic polynomial

$$\det(A - \lambda I) = (1 + \lambda)^2(5 - \lambda)$$

has roots $\lambda_1 = \lambda_2 = -1$ and $\lambda_3 = 5$. Computing eigenvectors we see that $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

and $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ form a basis for the eigenspace corresponding to $\lambda = -1$. We can apply the Gram-Schmidt process to obtain an orthonormal basis. Let

$$\mathbf{u}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{p}_2 = \mathbf{v}_2 \cdot \mathbf{u}_1 \mathbf{u}_1 = -\sqrt{2}\mathbf{u}_1 = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$$

$$\mathbf{v}_2 - \mathbf{p}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{u}_2 = \frac{\mathbf{v}_2 - \mathbf{p}_2}{\|\mathbf{v}_2 - \mathbf{p}_2\|} = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

The eigenspace corresponding to $\lambda_3 = 5$ is spanned by $\mathbf{v}_3 = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$ and we set

$$\mathbf{u}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$

Hence $A = UDU^T$ where

$$U = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3] = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{3} & -1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}, \quad D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

Th Eigenvectors belonging to distinct eigenvalues of a symmetric matrix are orthogonal. The eigenvalues of a symmetric matrix are real.

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