Lecture 6: 1.8-1.9 Linear Transformations.
A **transformation** (or **mapping** or **function**) $T:\mathbf{R}^n \to \mathbf{R}^m$ is a rule that for each $\mathbf{x} \in \mathbf{R}^n$ assigns a vector $T(\mathbf{x}) \in \mathbf{R}^m$, called the image of $\mathbf{x}$.
Matrix multiplication by an $m\times n$ matrix A gives a mapping $\mathbf{R}^n \ni \mathbf{x} \to A\mathbf{x} \in \mathbf{R}^m$:

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \to \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n \end{bmatrix}.$$

A **matrix transformation** $T(\mathbf{x}) = A\mathbf{x}$ is the simplest type of transformation.

Ex 1 $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \to \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$ **rotates** vectors an angle $\pi/2$ counterclockwise.

Ex 2 $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \to \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 \\ 3x_2 \end{bmatrix}$ **scales** vectors by a factor 3.

Ex 3 $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$ **projects** vectors onto the x_1 axis.

Ex 4 $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \to \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_1 \\ x_2 \end{bmatrix}$ **reflects** vectors in the x_2 axis.

Ex 5 $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is the **identity** map.

Ex 6 $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \to \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 3x_2 \\ x_2 \end{bmatrix}$ is called **shear**

If $T:\mathbf{R}^n \to \mathbf{R}^m$ is a transformation then the set $\mathbf{R}^n$ is called the **domain** of T and $\mathbf{R}^m$ is called the **codomain**. The set of all images $T(\mathbf{x})$ is called the **range** of T.
T is said to be **onto** if each $\mathbf{y} \in \mathbf{R}^m$ is the image $T(\mathbf{x})$ of at least one $\mathbf{x} \in \mathbf{R}^n$.
T is said to be **one-to-one** if each $\mathbf{y} \in \mathbf{R}^m$ is the image of at most one $\mathbf{x} \in \mathbf{R}^n$.
T is called **invertible** if its one-to-one and onto.
The transformations in Ex 1-2 and 4-6 are onto and one-to-one, but Ex 3 is not.

Ex 6 Define $T:\mathbf{R}^2 \to \mathbf{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$, where $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}$. Is T onto?

Sol The range of T is the span of the two column vectors of A, which is a plane in $\mathbf{R}^3$. Every vector outside this plane is not in the range and T, i.e. there is $\mathbf{b} \in \mathbf{R}^3$ such that $\mathbf{b} \neq T(\mathbf{x})$ for all $\mathbf{x} \in \mathbf{R}^2$ so T is not onto. We can see this by trying to solve $A\mathbf{x} = \mathbf{b}$:

$$\begin{bmatrix} 1 & 0 & b_1 \\ 2 & 1 & b_2 \\ 0 & 1 & b_3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & b_1 \\ 0 & 1 & b_2 - 2b_1 \\ 0 & 1 & b_3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & b_1 \\ 0 & 1 & b_2 - 2b_1 \\ 0 & 0 & b_3 - b_2 + 2b_1 \end{bmatrix}$$

which only has a solution if $b_3 - b_2 + 2b_1 = 0$. The range is the plane $b_3 - b_2 + 2b_1 = 0$.

Ex 7 Define $T:\mathbf{R}^3 \to \mathbf{R}^2$ by $T(\mathbf{x}) = A\mathbf{x}$, where $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. Is T one-to-one?

Sol $A\mathbf{x} = \mathbf{0}$ has nontrivial solutions since there are more variables than equations. Hence there are infinitely many points such that $T(\mathbf{x}) = 0$ so T is not one-to-one. Explicitly

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \quad \Leftrightarrow \quad \begin{aligned} x_1 &= 2x_3 \\ x_2 &= -x_3 \\ x_3 &= \text{free} \end{aligned}$$

A transformation is called **linear** if $T\big(\mathbf{x}+\mathbf{y}\big)=T(\mathbf{x})+T(\mathbf{y})$ and $T(\lambda\mathbf{x})=\lambda T(\mathbf{x})$, which can alternatively be summarized in $T(\lambda_1\mathbf{x}_1+\lambda_2\mathbf{x}_2)=\lambda_1 T(\mathbf{x}_1)+\lambda_2 T(\mathbf{x}_2)$.

Th If A is a $m\times n$ matrix then $T(\mathbf{x})=A\mathbf{x}$ is a linear transformation $\mathbf{R}^n\to\mathbf{R}^m$.
Pf $T(\mathbf{x}+\mathbf{y})=A(\mathbf{x}+\mathbf{y})=A\mathbf{x}+A\mathbf{y}=T(\mathbf{x})+T(\mathbf{y})$ and $T(\lambda\mathbf{x})=A(\lambda\mathbf{x})=\lambda A\mathbf{x}=\lambda T(\mathbf{x})$. (These properties follows from the definition of the matrix product $A\mathbf{x}$.)

Ex 8 $T(x_1,x_2)=(x_1+2,x_2+x_1)$ and $T(x_1,x_2)=(x_1+x_2^2,x_1)$ are not linear.

Ex 9 Let $\mathbf{e}_1=\begin{bmatrix}1\\0\end{bmatrix}$, $\mathbf{e}_2=\begin{bmatrix}0\\1\end{bmatrix}$, $\mathbf{w}_1=\begin{bmatrix}1\\0\\2\end{bmatrix}$, $\mathbf{w}_2=\begin{bmatrix}0\\1\\1\end{bmatrix}$, $\mathbf{c}=\begin{bmatrix}3\\2\end{bmatrix}$.
Suppose $T:\mathbf{R}^2\to\mathbf{R}^3$ is linear and $T(\mathbf{e}_1)=\mathbf{w}_1$, $T(\mathbf{e}_2)=\mathbf{w}_2$. Find the images of $\mathbf{c}$.
Sol We can write $\mathbf{c}=\begin{bmatrix}3\\2\end{bmatrix}=3\begin{bmatrix}1\\0\end{bmatrix}+2\begin{bmatrix}0\\1\end{bmatrix}=3\mathbf{e}_1+2\mathbf{e}_2$. Since T is linear it follows that

$$T(\mathbf{c})=T(3\mathbf{e}_1+2\mathbf{e}_2)=3T(\mathbf{e}_1)+2T(\mathbf{e}_2)=3\mathbf{w}_1+2\mathbf{w}_2=3\begin{bmatrix}1\\0\\2\end{bmatrix}+2\begin{bmatrix}0\\1\\1\end{bmatrix}=\begin{bmatrix}3\\2\\8\end{bmatrix}.$$

In general if $\mathbf{x}\in\mathbf{R}^n$ we can write

$$\mathbf{x}=\begin{bmatrix}x_1\\\vdots\\x_n\end{bmatrix}=x_1\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}+\cdots+x_n\begin{bmatrix}0\\\vdots\\0\\1\end{bmatrix}=x_1\mathbf{e}_1+\cdots+x_n\mathbf{e}_n$$

and by repeated use of linearity

$$T(\mathbf{x})=T\big(x_1\mathbf{e}_1+\cdots+x_n\mathbf{e}_n\big)=x_1T(\mathbf{e}_1)+\cdots+x_nT(\mathbf{e}_n)$$

This is the matrix multiplication of the matrix $[\,T(\mathbf{e}_1)\quad\cdots\quad T(\mathbf{e}_n)\,]$ and $\mathbf{x}$ so

Th If T is a linear transformation $\mathbf{R}^n\to\mathbf{R}^m$ then $T(\mathbf{x})=A\mathbf{x}$, where $A=\big[\mathbf{a}_1\cdots\mathbf{a}_n\big]$ is the matrix whose j:th column is $\mathbf{a}_j=T(\mathbf{e}_j)$ and $\mathbf{e}_j$ is the vector with 1 in the jth place and 0 otherwise.

Ex 10 Rotating by an angle θ counterclockwise in the plane is a linear transformation.
Pf That $T(\alpha\mathbf{x})=\alpha T(\mathbf{x})$ says that it is the same thing to first multiply by α and then rotate as it is to first rotate and then multiply by α, which is clear from the geometric definition of scalar multiplication. That $T(\mathbf{x}+\mathbf{y})=T(\mathbf{x})+T(\mathbf{y})$ says that it the same thing to first add the vectors and then rotate the sum as it is to first rotate them and then add the results. This follows from the geometric definition of addition with the parallelogram law, since rotating the whole parallelogram by an angle θ rotates all its sides by the same angle.

Ex 10 Rotating the angle θ is given by multiplying by the matrix $Q=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$.

Pf Since it is a linear it follows that $Q=[\mathbf{q}_1\ \mathbf{q}_2]$, where $\mathbf{q}_1=T(\mathbf{e}_1)$ and $\mathbf{q}_2=T(\mathbf{e}_2)$. The rotation of $\mathbf{e}_1$ by an angle θ is the vector $\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}$ and rotation of $\mathbf{e}_2$ is $\begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}$.

Ex 12 Suppose that $T:\mathbf{R}^2\to\mathbf{R}^3$ is the map $T(x_1,x_2)=x_1-2x_2,4x_1,3x_1+2x_2)$. Find the matrix for T. **Sol** $A=[\,T(\mathbf{e}_1)\quad T(\mathbf{e}_2)\,]=\begin{bmatrix}1 & -2\\ 4 & 4\\ 3 & 2\end{bmatrix}$.