Lecture 7: 2.1 Matrix algebra. An $m \times n$ matrix A is a collection of numbers (called entries) $[\, a_{ij} \,]_{1 \le i \le m,\, 1 \le j \le n}$

$$\begin{array}{cc} & j \text{ th } \textbf{column} \\ A = & \begin{bmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ & & \vdots & & \\ a_{i1} & \dots & a_{ij} & \dots & a_{in} \\ & & \vdots & & \\ a_{m1} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix} \quad i \text{ th } \textbf{row} \end{array}$$

we often write it in terms of column vectors: $A = [\, \mathbf{a}_1 \quad \cdots \quad \mathbf{a}_n \,]$, $\qquad \mathbf{a}_j = \begin{bmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{bmatrix}$
or just $A = [\, a_{ij} \,]$. We will use the notation $(A)_{ij}$ to denote the (i,j)**th entry** a_{ij}.

If A and B are two $m \times n$ matrices the **sum** $A + B$ is the $m \times n$ matrix whose (i,j) the entry is the sum of the entries at the corresponding places in A and B

$$(A + B)_{ij} = (A)_{ij} + (B)_{ij}$$

Ex $\qquad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 1+0 & 2+1 \\ 3-3 & 4+2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 6 \end{bmatrix}$

The **scalar multiplication** λA is the $m \times n$ matrix whose (i,j) the entry is the product of λ and the entry at the corresponding place in A

$$(\lambda A)_{ij} = \lambda (A)_{ij}$$

Ex $\qquad 3 \cdot \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 3 \cdot 1 & 3 \cdot 2 & 3 \cdot 3 \\ 3 \cdot 4 & 3 \cdot 5 & 3 \cdot 6 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 9 \\ 12 & 15 & 18 \end{bmatrix}$

The zero matrix is

$$0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The addition satisfy $A + B = B + A$, $A + 0 = 0$ etc. since we are just adding the entries at the same place and this is true for real numbers.

Composition of linear maps and matrix product. One is often interested in **composing** transformations i.e. one first perform one transformations and then another to the result of the first, e.g. if we first rotate a vector and then scale the resulting vector or rotate it again. If the transformations involved are linear then the composite transformation is also linear and one would like to calculate the matrix for the composite transformation from those of the involved transformations. Suppose that A is an $m \times n$ matrix and B is an $p \times n$ matrix. Then multiplication by B defines a map $\mathbf{R}^p \ni \mathbf{x} \to B\mathbf{x} \in \mathbf{R}^n$ and multiplication by A defines a map $\mathbf{R}^n \ni \mathbf{y} \to A\mathbf{y} \in \mathbf{R}^m$ and multiplication by first B and then A

$$\mathbf{x} \xrightarrow{\text{multiply by } B} B\mathbf{x} \xrightarrow{\text{multiply by } A} A(B\mathbf{x})$$

defines a map $\mathbf{R}^p \ni \mathbf{x} \to A(B\mathbf{x}) \in \mathbf{R}^m$. We want to define the matrix product AB to be the $m \times p$ matrix that represents this map so that $(AB)\mathbf{x} = A(B\mathbf{x})$:

$$\mathbf{x} \xrightarrow{\qquad\text{multiply by } AB\qquad} (AB)\mathbf{x} = A(B\mathbf{x})$$

Another way to formulate this is to say that we want the matrix multiplication to be defined so it is **associative**, i.e. $(AB)\mathbf{x} = A(B\mathbf{x})$, it shouldn't matter if you first calculate $B\mathbf{x}$ and then $A(B\mathbf{x})$ or if you first calculate AB and then $(AB)\mathbf{x}$.
Question: How can we find the matrix for the composition of linear maps if we know the matrices for the maps themselves? Let us calculate $A(B\mathbf{x})$ and see

$$B\mathbf{x} = [\,\mathbf{b}_1 \cdots \mathbf{b}_p\,] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = B\mathbf{x} = x_1\mathbf{b}_1 + \cdots + x_p\mathbf{b}_p.$$

and hence by linearity

$$A(B\mathbf{x}) = A\big(x_1\mathbf{b}_1 + \cdots + x_p\mathbf{b}_p\big) = x_1 A\mathbf{b}_1 + \cdots + x_p A\mathbf{b}_p = [\,A\mathbf{b}_1 \cdots A\mathbf{b}_p\,] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

can be interpreted as the matrix product of the $m \times p$ matrix with columns $A\mathbf{b}_1, \ldots, A\mathbf{b}_p$ and the column vector $\mathbf{x}$. Since we already know how to calculate $A\mathbf{b}_j$ where $\mathbf{b}_j$ is a column vector this allows us to define the **matrix multiplication** to be

$$AB = [\,A\mathbf{b}_1 \quad \cdots \quad A\mathbf{b}_p\,]$$

and we have achieved that $(AB)\mathbf{x} = A(B\mathbf{x})$. (That the columns of the matrix of the transformation $\mathbf{x} \to A(B\mathbf{x})$ are $A(B\mathbf{e}_j) = A\mathbf{b}_j$ also follows from section 1.9.)

For practical calculations by hand it is more efficient to use the alternative **row-column rule** to compute the (i,j)th entry of AB as the dot product between the ith row of $A = [\,a_{ij}\,]$ and jth column of $B = [\,b_{ij}\,]$:

$$(AB)_{ij} = a_{i1}b_{1j} + \cdots + a_{in}b_{nj}$$

$$\begin{array}{ccccc} i\text{ th }\textbf{row} & \begin{bmatrix} \\ a_{i1} \cdots a_{in} \\ \\ \end{bmatrix} \begin{bmatrix} b_{1j} \\ \vdots \\ b_{nj} \end{bmatrix} & = & \begin{bmatrix} \vdots \\ \cdots (AB)_{ij} \cdots \\ \vdots \end{bmatrix} & i\text{ th }\textbf{row} \\ & \qquad\qquad j\text{ th }\textbf{column} & & j\text{ th }\textbf{column} & \end{array}$$

Ex Let $A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}$. Find AB

$$\textbf{Sol} \quad A\mathbf{b}_1 = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1\cdot(-1) + 2\cdot 1 \\ 0\cdot(-1) + (-1)\cdot 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

$$A\mathbf{b}_2 = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1\cdot 1 + 2\cdot 2 \\ 0\cdot 1 + (-1)\cdot 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

Hence

$$AB = A\begin{bmatrix} \mathbf{b}_1 \mathbf{b}_2 \end{bmatrix} = \begin{bmatrix} A\mathbf{b}_1 A\mathbf{b}_2 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ -1 & -2 \end{bmatrix}$$

Alternatively using the row-column method

$$AB = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1(-1) + 2\cdot 1 & 1\cdot 1 + 2\cdot 2 \\ 0(-1) + (-1)1 & 0\cdot 1 + (-1)2 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ -1 & -2 \end{bmatrix}$$

Alternatively one can also write this as

$$\begin{array}{cc} & \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix} \\ AB = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} & \begin{bmatrix} 1 & 5 \\ -1 & -2 \end{bmatrix} \end{array}$$

Quest When is the product AB of an $m \times n$ matrix A and a $p \times q$ matrix B defined?
Question Is matrix multiplication **commutative**, i.e. is $AB = BA$?
Why do people expect things to be commutative in math when they are not commutative in real life? It is not the same thing to first put on the shoes and then the socks as it is to first put on the socks and then the shoes?
What if A is a 2×3 and B is 3×2? Are AB and BA defined?
Is it the same to first rotate and then reflect as it is to first reflect and then rotate?
Ex Let $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ be the matrix of rotation $\frac{\pi}{2}$ counterclockwise and $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ be the matrix of reflection in the x_1 axis. Find AB and BA. Interpret geometrically.
Sol

$$\begin{array}{cc} & \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ AB = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array} \qquad \begin{array}{cc} & \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\ BA = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} & \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \end{array}$$

Matrix multiplication need not be commutative, i.e. in general $AB \neq BA$.
Question Given examples of nonzero matrices such that $AB = 0$.
If you first project on the x_1 axis and then on the x_2 axis the result is $\mathbf{0}$.
The identity matrix is $I = [\,\delta_{ij}\,]$, where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$:

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \text{in case } 4 \times 4.$$

We have $AI = IA = A$ for any matrix A if I has the right size.
The transpose A^T is the matrix with rows and columns interchanged, $(A^T)_{ij} = (A)_{ji}$

Ex If $A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 0 & -1 \\ 4 & 5 & 2 \end{bmatrix}$ then $A^T = \begin{bmatrix} 1 & -2 & 4 \\ 2 & 0 & 5 \\ 3 & -1 & 2 \end{bmatrix}$.
We have e.g. $(AB)^T = B^T A^T$, $(A + B)^T = A^T + B^T$.