Lecture 8: 2.2 The inverse of a matrix.

The inverse of a real number $a \neq 0$ is a number denoted a^{-1} such that

$$a \cdot a^{-1} = 1$$

An $n{\times}n$ matrix A is said to be **invertible** if there is an $n{\times}n$ matrix A^{-1} such that

$$A^{-1}A = AA^{-1} = I \tag{2.2.1}$$

where I is the identity matrix. The matrix A^{-1} called the **inverse** of A is unique. In fact if $BA = AB = I$ then $B = BI = B(AA^{-1}) = (BA)A^{-1} = IA^{-1} = A^{-1}$. Not all $n \times n$ matrices are invertible. A matrix which is not invertible is called **singular**. An invertible matrix is called **nonsingular**.

Th If A is invertible then the equations $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.
Pf If A is invertible then $A^{-1}\mathbf{b}$ is a solution to the system $A\mathbf{x} = \mathbf{b}$. In fact,

$$A(A^{-1}\mathbf{b}) = (AA^{-1})\mathbf{b} = I\,\mathbf{b} = \mathbf{b}.$$

To see that it is the only solution to $A\mathbf{x} = \mathbf{b}$ we multiply both sides by A^{-1} to get

$$A^{-1}A\mathbf{x} = A^{-1}\mathbf{b},$$

and since

$$A^{-1}A\mathbf{x} = I\mathbf{x} = \mathbf{x},$$

it follows that

$$\mathbf{x} = A^{-1}\mathbf{b}.$$

Th Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad-bc \neq 0$ then A is invertible and $A^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.
If $ad - bc = 0$ then A is not invertible.
Pf If $ad - bc \neq 0$ its easy to check that $AA^{-1} = A^{-1}A = I$:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}.$$

If $ad - bc = 0$ then (a, b) and (c, d) are proportional and the system

$$\begin{aligned} ax_1 + bx_2 &= b_1 \\ cx_1 + dx_2 &= b_2 \end{aligned}$$

does not have a unique solution so the conclusion in the preceding theorem is wrong and hence the assumption must be wrong, i.e. A is not invertible.
Ex Solve the system

$$\begin{aligned} -7x_1 + 3x_2 &= 2 \\ 5x_1 - 2x_2 &= 1 \end{aligned}$$

$$A = \begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix},\ A^{-1} = \frac{1}{7{\cdot}2 - 3{\cdot}5}\begin{bmatrix} -2 & -3 \\ -5 & -7 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix} \text{ so } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 17 \end{bmatrix}$$

An **inverse** of a transformation $\mathbf{x} \to T(\mathbf{x})$ is a transformation which takes you back $T(\mathbf{x}) \to \mathbf{x}$. The condition $A^{-1}A = I$ says that the inverse of the linear transformation $\mathbf{x} \to A\mathbf{x}$ is the linear transformation $\mathbf{y} \to A^{-1}\mathbf{y}$. In fact, if we compose $\mathbf{x} \to A\mathbf{x}$ with $\mathbf{y} \to A^{-1}\mathbf{y}$ we get $\mathbf{x} \to A\mathbf{x} \to A^{-1}(A\mathbf{x}) = (AA^{-1})\mathbf{x} = I\mathbf{x} = \mathbf{x}$.
Question What is the inverse of a scaling by a factor 3 and what is its matrix? What is the inverse of a rotation counterclockwise angle $\pi/2$ and what is its matrix?

$(AB)^{-1} = B^{-1}A^{-1}$, $(A^{-1})^{-1} = A$ as is clear from the composition of transformations.

Question: How can we calculate the inverse of a matrix?
If we can solve $A\mathbf{x} = \mathbf{y}$ for any $\mathbf{y}$ we will get the inverse $\mathbf{x} = A^{-1}\mathbf{y}$.

Ex Find the inverse of $A = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

Sol We perform row operations to solve the system $A\mathbf{x} = \mathbf{y}$:

$$\begin{cases} x_1 = y_1 \\ -3x_1 + x_3 = y_2 \\ x_2 = y_3 \end{cases} \Leftrightarrow \begin{cases} x_1 = y_1 \\ x_3 = 3y_1 + y_2 \\ x_2 = y_3 \end{cases} \Leftrightarrow \begin{cases} x_1 = y_1 \\ x_2 = y_3 \\ x_3 = 3y_1 + y_2 \end{cases}$$

adding three times the first equation to the second and then switching the second and the third equations. The system on the right is $\mathbf{x} = A^{-1}\mathbf{y}$ so we must have that $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix}$. Its easy to check that $AA^{-1} = I$.

The calculations above can be performed without writing out the variables as row operations directly to the augmented matrix $[\,A \;\; I\,]$;

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{matrix} \\ (2)+3(1) \\ \\ \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{matrix} \\ (3) \\ (2) \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 & 1 & 0 \end{bmatrix}$$

We have found an algorithm for determining if A is invertible and finding the inverse: Calculate the reduced row echelon form of the augmented matrix $[\,A \;\; I\,]$. If it is of the form $[\,I \;\; B\,]$ then A is invertible and $A^{-1} = B$. Otherwise A is not invertible.

One can also prove that this works multiplying by **elementary** matrices which correspond to elementary row operations. Let $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

Multiplying by E_1 adds 3 times row one to row two:

$E_1A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

Multiplying by E_2 switches row two and row three:

$E_2(E_1A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

Hence

$$E_2E_1A = I$$

and multiplying both sides by A^{-1} to the right gives since $AA^{-1} = I$ and $IA^{-1} = A^{-1}$:

$$E_2E_1I = A^{-1}$$

Hence a sequence of elementary row operations that reduce A to I reduce I to A^{-1}. This argument assumed that A was invertible, but it also follows since each elementary matrix is invertible since the row operations are reversible and hence multiplying by the inverse of the elementary matrices gives $A = E_1^{-1}E_2^{-1}$ so A is invertible since its a product of invertible matrices.

2.3 Characterizations of Invertible Matrices.

Question: What conditions are equivalent to that a matrix is invertible?

Th Let A be a given $n \times n$ matrix. Then the following are equivalent:

a) A is invertible.
b) A is row equivalent to I.
c) A has n pivot positions
d) The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
e) The columns of A are linearly independent.
f) The linear transformation $\mathbf{x} \to A\mathbf{x}$ is one-to-one.
g) The equations $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$.
h) The columns of A span $\mathbf{R}^n$.
i) The linear transformation $\mathbf{x} \to A\mathbf{x}$ is onto.
j) There is an $n \times n$ matrix C such that $CA = I$.
k) There is an $n \times n$ matrix D such that $AD = I$.
l) A^T is invertible.

Let us prove that (g) implies (a). If (g) is true then A must have n pivots and be row equivalent to the identity or else there would be some $\mathbf{b}$ for which it is inconsistent. In fact if R is the reduced row echelon form of A and if the last row of R has all zeros then the system $R\mathbf{x} = \mathbf{e}_n$, where $\mathbf{e}_n$ has 1 in the last place, is inconsistent. Hence if we perform the reverse of the row operations we get a system $A\mathbf{x} = \mathbf{b}$ which is inconsistent. But if its row equivalent to the identity then the same sequence of row operations reduce the identity to A^{-1}.

Let us also prove that (j) implies (a). If (j) holds then we can multiply $A\mathbf{x} = \mathbf{0}$ with C to obtain that $CA\mathbf{x} = C\mathbf{0}$ and hence by (j) $\mathbf{x} = \mathbf{0}$ so (d) holds. But if (d) holds there are no free variables so A has n pivot positions. But then the reduced row echelon form is the identity matrix so (b) holds. But using the algorithm in the previous section we can conclude that the same row operations that reduce A to I also reduce I to A^{-1}. Hence (a) holds.